Linear algebraic equations, Elimination of unknowns, Gauss Elimination, Techniques for improving solutions [1-5]

References:

1. Chapra S.C. and Canale R.P. "Numerical Methods for Engineers", Sixth Edition,McGraw Hill, International Edition 2010.

2. Chapra S.C. and Canale R. P. "Yazılım ve programlama Uygulamalarıyla Mühendisler için Sayısal Yöntemler" 4.Basımdan Çevirenler: Hasan Heperkan ve Uğur Kesgin 2003.

3.Chapra S.C. "Applied Numerical Methods with MATLAB for engineers and Scientists" Third Edition, McGraw Hill, International Edition 2012.

4. Mathews J.H. and Fınk K.D. "Numerical Methods using MATLAB", Fourth Edition, Pearson P. Hall, International Edition 2004.

5. Fausett L.V. "Applied Numerical Analysis Using MATLAB, Second Edition, PearsonP. Hall, International Edition, 2008.

6. Gilat A. And Subramaniam V. "Numerical Methods, An introduction with Applications Using MATLAB", Second Edition,John Wiley and Sons. Inc. 2011.

Gauss elimination

$$x_1 + x_2 + x_3 = 11$$

$$x_1 - 2x_2 + 2x_3 = 4$$

$$x_1 + x_2 - x_3 = 1$$

$$\begin{bmatrix} 1 & 1 & 1 & 11 \\ 1 & -2 & 2 & 4 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

Multiply the first equation by (1/1) and subtract the result from the second equation. (Reduction of the x_1 term from the second row.)

$$[1 \quad -2 \quad 2 \quad 4] - \left(\frac{1}{1}\right)[1 \quad 1 \quad 1 \quad 11] = [0 \quad -3 \quad 1 \quad -7]$$

Multiply the first equation by (1/1) and subtract the result from the third equation. (Reduction of the x_1 term from the third row.)

$$[1 \quad 1 \quad -1 \quad 1] - \left(\frac{1}{1}\right)[1 \quad 1 \quad 1 \quad 11] = [0 \quad 0 \quad -2 \quad -10]$$

After these operations the set of equations is;

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ -7 \\ -10 \end{Bmatrix}$$

We can now solve these equations by back-substitution.

$$-2x_3 = -10 \qquad \rightarrow \qquad x_3 = 5$$

$$-3x_2 + x_3 = -7 \qquad \rightarrow \qquad x_2 = 4$$

$$x_1 + x_2 + x_3 = 11 \qquad \rightarrow \qquad x_1 = 2$$

MATLAB commands to solve the linear algebraic equation set

$$5x_1 - 0.2x_2 - 0.8x_3 = 4.86$$
$$0.2x_1 + 9x_2 - 0.9x_3 = -58.02$$
$$0.4x_1 - 0.3x_2 + 12x_3 = 60$$

```
>> A=[5  -0.2  -0.8;0.2  9  -0.9;  0.4  -0.3  12];
>> B=[4.86; -58.02; 60];
>> x=A\B
```

or

```
>> x=inv(A)*B
```

Gauss elimination method to solve linear system of equations [A]{X}={B}

$$A = \begin{bmatrix} 49 & -7 & 14 \\ -7 & 26 & 3 \\ 14 & 3 & 21 \end{bmatrix}, X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, B = \begin{Bmatrix} 126 \\ -53 \\ -3 \end{Bmatrix}$$

$$[AB] = \begin{bmatrix} 49 & -7 & 14 & 126 \\ -7 & 26 & 3 & -53 \\ 14 & 3 & 21 & -3 \end{bmatrix}$$

Multiply the first row by (-7/49) and subtract the result from the second row. (Reduction of the x_1 term from the second row.)

$$[-7 \quad 26 \quad 3 \quad -53] - \left(\frac{-7}{49}\right)[49 \quad -7 \quad 14 \quad 126] = [0 \quad 25 \quad 5 \quad -35]$$

$$[AB] = \begin{bmatrix} 49 & -7 & 14 & 126 \\ 0 & 25 & 5 & -35 \\ 14 & 3 & 21 & -3 \end{bmatrix}$$

Multiply the first row by (14/49) and subtract the result from the third row. (Reduction of the x_1 term from the third row.)

$$[14 \;\; 3 \;\; 21 \;\; -3] - \left(\frac{14}{49}\right)[49 \;\; -7 \;\; 14 \;\; 126] = [0 \;\; 5 \;\; 17 \;\; -39]$$

$$[AB] = \begin{bmatrix} 49 & -7 & 14 & 126 \\ 0 & 25 & 5 & -35 \\ 0 & 5 & 17 & -39 \end{bmatrix}$$

Multiply the second row by (5/25) and subtract the result from the third row. (Reduction of the x_2 term from the third row.)

$$[0 \;\; 5 \;\; 17 \;\; -39] - \left(\frac{14}{49}\right)[0 \;\; 25 \;\; 5 \;\; -35] = [0 \;\; 0 \;\; 16 \;\; -32]$$

$$\begin{bmatrix} 49 & -7 & 14 \\ 0 & 25 & 5 \\ 0 & 0 & 16 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 126 \\ -35 \\ -32 \end{Bmatrix}$$

We can now solve these equations by back-substitution.

$$16x_3 = -32 \quad \rightarrow \quad x_3 = -2$$

$$25x_2 + 5x_3 = -35 \quad \rightarrow \quad x_2 = -1$$

$$49x_1 - 7x_2 + 14x_3 = 126 \quad \rightarrow \quad x_1 = 3$$

Gauss Elimination to solve

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6 \end{aligned}$$

$$x_1 + 2x_2 + x_3 + 4x_4 = 13 \quad (1)$$
$$2x_1 + 0x_2 + 4x_3 + 3x_4 = 28 \quad (2)$$
$$4x_1 + 2x_2 + 2x_3 + x_4 = 20 \quad (3)$$
$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6 \quad (4)$$

In order to obtain upper triangular matrix, we need to eliminate x_1, x_2 and x_3 from the set of equation

Step 1: Forward elimination for x_1

$$2 / \; x_1 + 2x_2 + x_3 + 4x_4 = 13 \quad (1)$$

$$2x_1 + 0x_2 + 4x_3 + 3x_4 = 28 \quad (2)$$
$$\underline{2x_1 + 4x_2 + 2x_3 + 8x_4 = 26 \quad (1)}$$
$$-4x_2 + 2x_3 - 5x_4 = 2 \quad (5) \qquad x_1 \textit{ eliminated}$$

Step 2: Forward elimination for x_1

$4/x_1 + 2x_2 + x_3 + 4x_4 = 13$ **(1)**

$4x_1 + 2x_2 + 2x_3 + x_4 = 20$ **(3)**

$4x_1 + 8x_2 + 4x_3 + 16x_4 = 52$ **(1)**

$-6x_2 - 2x_3 - 15x_4 = -32$ **(6)** ***x_1 eliminated***

Step 3: Forward elimination for x_1

$-3/x_1 + 2x_2 + x_3 + 4x_4 = 13$ **(1)**

$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$ **(4)**

$-\mathbf{3}x_1 - 6x_2 - 3x_3 - 12x_4 = -39$ **(1)**

$7x_2 + 6x_3 + 14x_4 = 45$ **(7)** ***x_1 eliminated***

Step 4: Forward elimination for x_2

$$\frac{6}{4} \;/\; -4x_2 + 2x_3 - 5x_4 = 2 \qquad \textbf{(5)}$$

$$-6x_2 - 2x_3 - 15x_4 = -32 \quad \textbf{(6)}$$
$$-6x_2 + 3x_3 - 7.5x_4 = 3 \quad \textbf{(5)}$$

$$-5x_3 - 7.5x_4 = -35 \quad \textbf{(8)}$$ x_2 ***eliminated***

Step 5: Forward elimination for x_2

$$-\frac{7}{4} \;/\; -4x_2 + 2x_3 - 5x_4 = 2 \qquad \textbf{(5)}$$

$$7x_2 + 6x_3 + 14x_4 = 45 \quad \textbf{(7)}$$
$$7x_2 - 3.5x_3 + 8.75x_4 = -3.5 \quad \textbf{(5)}$$

$$9.5x_3 + 5.25x_4 = 48.5 \quad \textbf{(9)}$$ x_2 ***eliminated***

Step 6: Forward elimination for x_3

$$-\frac{9.5}{5} \quad / -5x_3 - 7.5x_4 = -35 \quad \textbf{(8)}$$

$$9.5x_3 + 5.25x_4 = 48.5 \quad \textbf{(9)}$$

$$\underline{9.5x_3 + 14.25x_4 = 66.5 \quad \textbf{(8)}}$$

$$-9x_4 = -18 \qquad \textbf{(10)} \qquad x_3 \; \textit{eliminated}$$

Therefore the upper triangular matrix:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13 \qquad \textbf{(1)}$$

$$-4x_2 + 2x_3 - 5x_4 = 2 \qquad \textbf{(5)}$$

$$-5x_3 - 7.5x_4 = -35 \qquad \textbf{(8)}$$

$$-9x_4 = -18 \qquad \textbf{(10)}$$

Back substitution:

$-9x_4 = -18 \quad x_4 = \mathbf{2}$

$-5x_3 - 7.5x_4 = -35 \qquad -5x_3 - 15 = -35 \qquad x_3 = \mathbf{4}$

$-4x_2 + 2x_3 - 5x_4 = 2 \qquad -4x_2 + 8 - 10 = 2 \qquad x_2 = \mathbf{-1}$

$x_1 + 2x_2 + x_3 + 4x_4 = 13 \qquad x_1 - 2 + 4 + 8 = 13 \qquad x_1 = \mathbf{3}$