LU decomposition, Matrix Inversion [1-6]

References:

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Employ LU decomposition to determine the matrix inverse

$$[U] = \begin{bmatrix} 2 & -6 & -1 \\ 0 & -10 & 5.5 \\ 0 & 0 & -18.65 \end{bmatrix} \quad \text{and} \quad [L] = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -4 & 2.3 & 1 \end{bmatrix}$$

First step:

$$[L] * \{D\} = \{b\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -4 & 2.3 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$d_1 = 1$$

$$-1.5d_1 + d_2 = 0$$

$$-4d_1 + 2.3d_2 + d_3 = 0 \quad \rightarrow \quad d_2 = 1.5 \text{ and } d_3 = 0.55. \text{ So; } \{D\} = \begin{Bmatrix} 1 \\ 1.5 \\ 0.55 \end{Bmatrix}$$

$$[U] * \{X\} = \{D\}$$

$$\begin{bmatrix} 2 & -6 & -1 \\ 0 & -10 & 5.5 \\ 0 & 0 & -18.65 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.5 \\ 0.55 \end{Bmatrix}$$

$$2x_1 - 6x_2 - x_3 = 1$$
$$-10x_1 + 5.5x_3 = 1.5 \quad \rightarrow x_1 = \text{-0.0134};\ x_2 = \text{-0.1662};\ x_3 = \text{-0.0295}$$
$$-18.65x_3 = 0.55$$

The first column of $A^{-1} = \begin{Bmatrix} -0.0134 \\ -0.1662 \\ -0.0295 \end{Bmatrix}$

<u>Second step:</u>

$$[L] * \{D\} = \{b\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -4 & 2.3 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$d_1 = 0$$
$$-1.5d_1 + d_2 = 1$$
$$-4d_1 + 2.3d_2 + d_3 = 0 \quad \rightarrow \quad d_2 = 1 \text{ and } d_3 = -2.3. \text{ So; } \{D\} = \begin{Bmatrix} 0 \\ 1 \\ -2.3 \end{Bmatrix}$$

$$[U] * \{X\} = \{D\}$$

$$\begin{bmatrix} 2 & -6 & -1 \\ 0 & -10 & 5.5 \\ 0 & 0 & -18.65 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ -2.3 \end{Bmatrix}$$

$2x_1 - 6x_2 - x_3 = 0$

$-10x_1 + 5.5x_3 = 1 \quad \rightarrow x_1$= -0.0349; x_2= -0.0322; x_3= 0.1233

$-18.65x_3 = -2.3$

The second column of A^{-1} = $\begin{Bmatrix} -0.0349 \\ -0.0322 \\ 0.1233 \end{Bmatrix}$

Third step:

$$[L] * \{D\} = \{b\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -4 & 2.3 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$d_1 = 0$

$-1.5d_1 + d_2 = 0$

$-4d_1 + 2.3d_2 + d_3 = 1 \quad \rightarrow \quad d_2 = 0$ and $d_3 = 1$ So; $\{D\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

$$[U] * \{X\} = \{D\}$$

$$\begin{bmatrix} 2 & -6 & -1 \\ 0 & -10 & 5.5 \\ 0 & 0 & -18.65 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\begin{aligned} 2x_1 - 6x_2 - x_3 &= 0 \\ -10x_1 + 5.5x_3 &= 0 \quad \rightarrow x_1 = -0.1153;\ x_2 = -0.0295;\ x_3 = -0.0536 \\ -18.65x_3 &= 1 \end{aligned}$$

The third column of $A^{-1} = \begin{Bmatrix} -0.1153 \\ -0.0295 \\ -0.0536 \end{Bmatrix}$

Therefore: $[A^{-1}] = \begin{bmatrix} -0.0134 & -0.0349 & -0.1153 \\ -0.1662 & -0.0322 & -0.0295 \\ -0.0295 & 0.1233 & -0.0536 \end{bmatrix}$

LU decomposition to determine the matrix inverse

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

$$\begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

Forward elimination:

Multiply 1 st row by $\frac{-3}{10}$ and subtracted from 2 nd row

$$[-3 \quad -6 \quad 2] - (\frac{-3}{10})[10 \quad 2 \quad -1] = [0 \quad -5.4 \quad 1.7]$$

Multiply 2 nd row by $\frac{1}{10}$ and subtracted from 3rd row

$$[1 \quad 1 \quad 5] - (\frac{1}{10})[10 \quad 2 \quad -1] = [0 \quad 0.8 \quad 5.1]$$

Multiply 2 nd row by $\frac{-0.8}{5.4}$ and subtracted from 3rd row

$$[0 \quad 0.8 \quad 5.1] - (\frac{-0.8}{5.4})[0 \quad -5.4 \quad 1.7] = [0 \quad 0 \quad 5.3518]$$

After forward elimination the **following upper triangular matrix** can be obtained:

$$U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3518 \end{bmatrix}$$

The factors employed to obtain the upper triangular matrix can be assemled into a loer triangular matrix:

$$f_{21} = \frac{-3}{10} \qquad f_{31} = \frac{1}{10}$$

$$a_{32}' \text{ eliminated by } f_{32} = \frac{0.8}{-5.4}$$

Lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.1481 & 1 \end{bmatrix}$$

Matrix inversion:

$$L=\begin{bmatrix}1 & 0 & 0\\ -0.3 & 1 & 0\\ 0.1 & -0.1481 & 1\end{bmatrix} \qquad U=\begin{bmatrix}10 & 2 & -1\\ 0 & -5.4 & 1.7\\ 0 & 0 & 5.3518\end{bmatrix}$$

$[L][d]=[b]$ b is the unit vector

For the first column of inverse matrix of A, $b=\begin{bmatrix}1\\0\\0\end{bmatrix}$ unit matrix is used.

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.1481 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$d_1=1$ $d_2=0.3$ $d_3=-0.05557$

$$[U][x]=[d]$$

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3518 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3 \\ -0.05557 \end{bmatrix}$$

$x_3=-0.01038$ $x_2=-0.0588$ $x_1=0.1107$

For the second column of inverse matrix of A, $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ unit matrix is used.

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.1481 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$d_1=0$ $d_2=1$ $d_3=0.1481$

$$[U][x] = [d]$$

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3518 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0.1481 \end{bmatrix}$$

$x_1=0.03806$ $x_2=-0.1764$ $x_3=0.02767$

For the third column of inverse matrix of A, $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ unit matrix is used.

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.1481 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ $d_1=0$ $d_2=0$ $d_3=1$

$$[U][x] = [d]$$

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3518 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ $x_1=0.00692$ $x_2=0.0588$ $x_3=0.1868$

Inverse matrix

$$A^{-1} = \begin{bmatrix} 0.1107 & 0.03806 & 0.00692 \\ -0.0588 & -0.1764 & 0.0588 \\ -0.01038 & 0.02767 & 0.1868 \end{bmatrix}$$