Special Matrices [1-6]

References:

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Cholesky decomposition to the symmetric matrix

$$\begin{bmatrix} 8 & 20 & 15 \\ 20 & 80 & 50 \\ 15 & 50 & 60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 250 \\ 100 \end{Bmatrix}$$

For the first row (k=1)

$$l_{11} \quad \rightarrow \quad k = 1 \quad \rightarrow \quad l_{11} = \sqrt{a_{11} - \sum_{j=1}^{0} l_{kj}^2} \quad \rightarrow \quad l_{11} = \sqrt{a_{11}} = \sqrt{8} = 2.8284$$

For the second row (k=2)

$$l_{21} \quad \rightarrow \quad k = 2, \quad i = 1 \quad \rightarrow \quad l_{21} = \frac{a_{21} - \sum_{j=1}^{0} l_{ij} l_{kj}}{l_{11}} \quad \rightarrow \quad l_{21} = \frac{a_{21}}{l_{11}} = \frac{20}{2.8284} = 7.0711$$

$$l_{22} \quad \rightarrow \quad k = 2 \quad \rightarrow \quad l_{22} = \sqrt{a_{22} - \sum_{j=1}^{1} l_{kj}^2} \quad \rightarrow \quad l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{80 - 7.0711^2} = 5.4772$$

For the third row (k=3)

$$l_{31} \quad \rightarrow \quad k = 3, \quad i = 1 \quad \rightarrow \quad l_{31} = \frac{a_{31} - \sum_{j=1}^{0} l_{ij} l_{kj}}{l_{11}} \quad \rightarrow \quad l_{31} = \frac{a_{31}}{l_{11}} = \frac{15}{2.8284} = 5.3034$$

$$l_{32} \quad \rightarrow \quad k = 3, \quad i = 2 \quad \rightarrow \quad l_{32} = \frac{a_{32} - \sum_{j=1}^{1} l_{ij} l_{kj}}{l_{22}}$$

$$l_{32} = \frac{a_{32} - l_{21} l_{31}}{l_{22}} = \frac{50 - 7.0711 * 5.3034}{5.4772} = 2.2820$$

$$l_{33} \quad \rightarrow \quad k = 3 \quad \rightarrow \quad l_{33} = \sqrt{a_{33} - \sum_{j=1}^{2} l_{kj}^2} \quad \rightarrow \quad l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

$$l_{33} = \sqrt{60 - 5.3034^2 - 2.2820^2} = 5.1640$$

Thus the Cholesky decomposition yields;

$$L = \begin{bmatrix} 2.8284 & & \\ 7.0711 & 5.4772 & \\ 5.3034 & 2.2820 & 5.1640 \end{bmatrix}$$

Forward substitution:

$[L]\{D\} = \{B\}$

$$\begin{bmatrix} 2.8284 & & \\ 7.0711 & 5.4772 & \\ 5.3034 & 2.2820 & 5.1640 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 250 \\ 100 \end{Bmatrix} \quad \rightarrow \quad d_1 = 35.3557 \quad d_2 = 0.0007 \quad d_3 = -16.9456$$

Backward substitution:

$[L]^T\{X\} = \{D\}$

$$\begin{bmatrix} 2.8284 & 7.0711 & 5.3034 \\ & 5.4772 & 2.2820 \\ & & 5.1640 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 35.3557 \\ 0.0007 \\ -16.9456 \end{Bmatrix} \quad \rightarrow \quad X = \begin{Bmatrix} -3.2815 \\ 1.3673 \\ 15.2349 \end{Bmatrix}$$

Tridiagonal Solution with the Thomas Algorithm

$$\begin{bmatrix} 4.04 & -1 & & \\ -1 & 4.04 & -1 & \\ & -1 & 4.04 & -1 \\ & & -1 & 4.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 172.7408 \\ 188.35512 \\ 249.87552 \\ 519.7612 \end{Bmatrix}$$

First, the decomposition is implemented as;

$e_2 = -1 / 4.04 = -0.248$

$f_2 = 4.04 - (-0.246)(-1) = 3.792$

$e_3 = -1 / 3.792 = -0.264$

$f_3 = 4.04 - (-0.264)(-1) = 3.776$

$e_4 = -1 / 3.776 = -0.265$

$f_4 = 4.04 - (-0.265)(-1) = 3.775$

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \quad g_4 \end{bmatrix}$$

$$\begin{matrix} 4.04 & -1 & & \\ -0.248 & 3.792 & -1 & \\ & -0.264 & 3.776 & -1 \\ & & -0.265 & 3.775 \end{matrix}$$

Thus, the matrix has been transformed to ;

$$\begin{bmatrix} 4.04 & -1 & & \\ -0.248 & 3.792 & -1 & \\ & -0.264 & 3.776 & -1 \\ & & -0.265 & 3.775 \end{bmatrix}$$

and the LU decomposition is ;

$$[A] = [L][U] = \begin{bmatrix} 1 & & & \\ -0.248 & 1 & & \\ & -0.264 & 1 & \\ & & -0.265 & 1 \end{bmatrix} \begin{bmatrix} 4.04 & -1 & & \\ & 3.792 & -1 & \\ & & 3.776 & -1 \\ & & & 3.775 \end{bmatrix}$$

verify → $[L][U]$ to yield $[A]$.

The forward substitution is implemented as; $[L]\{R\} = \{B\}$

$$\begin{bmatrix} 1 & & & \\ -0.248 & 1 & & \\ & -0.264 & 1 & \\ & & -0.265 & 1 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 172.7408 \\ 188.3551 \\ 249.8756 \\ 519.7612 \end{Bmatrix} \rightarrow \begin{matrix} r_1 = 172.7408 \\ r_2 = 231.1948 \\ r_3 = 310.9110 \\ r_4 = 602.1526 \end{matrix}$$

the right-hand-side vector

$$\{R\} = \begin{Bmatrix} 172.7408 \\ 231.1948 \\ 310.9110 \\ 602.1526 \end{Bmatrix}$$

to perform back substitution $[U]\{T\} = \{R\}$ and obtain solution ;

$$\begin{bmatrix} 4.04 & -1 & & \\ & 3.792 & -1 & \\ & & 3.776 & -1 \\ & & & 3.775 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 172.7408 \\ 231.1948 \\ 310.9110 \\ 602.1526 \end{Bmatrix} \rightarrow \begin{matrix} T_4 = 159.511 \\ T_3 = 124.5821 \\ T_2 = 93.8230 \\ T_1 = 65.9811 \end{matrix}$$