Ordinary Differential Equations, Runge-Kutta Methods [1-6]

References:

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the classical fourth order Runge Kutta method

using a step size of h=0.5 with y(0)=0 from x=0 to 0.5.

$$\frac{dy}{dx} = 1 - 4xy$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

$x_i = 0, y_i = 0$

The slope at the beginning of the interval is computed as

$k_1 = f(0,0) = 1 - 4 * 0 * 0 = 1$

This value is used to compute a value of y and a slope at the midpoint.

$$x_i + \frac{1}{2}h = 0 + \frac{1}{2} * 0.5 = 0.25$$

$$y_i + \frac{1}{2}k_1 h = 0 + \frac{1}{2} * 1 * 0.5 = 0.25$$

$k_2 = f(0.25, 0.25) = 1 - 4 * 0.25 * 0.25 = 0.75$

This slope is used to compute another value of y and another slope at the midpoint.

$$x_i + \frac{1}{2}h = 0 + \frac{1}{2} * 0.5 = 0.25$$

$$y_i + \frac{1}{2}k_2 h = 0 + \frac{1}{2} * 0.75 * 0.5 = 0.1875$$

$$k_3 = f(0.25, 0.1875) = 1 - 4 * 0.25 * 0.1875 = 0.8125$$

This slope is used to compute a value of y and a slope at the end of the interval.

$$x_i + h = 0 + 0.5 = 0.5$$

$$y_i + k_3 h = 0 + 0.8125 * 0.5 = 0.4063$$

$$k_4 = f(0.5, 0.4063) = 1 - 4 * 0.5 * 0.4063 = 0.1874$$

Finally, the four slope estimates are combined to yield an average slope.

$$\phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(1 + 2 * 0.75 + 2 * 0.8125 + 0.1874)$$

$$\phi = 0.7187$$

$$y(0.5) = y(0) + \phi h = 0 + 0.7187 * 0.5$$

$$y(0.5) = 0.3594$$

the fourth-order Runge Kutta methods

$$\frac{dy}{dx} = 2y + x + 4 \qquad y(0) = 1$$

to solve y(0.2)

h=0.2

$$\frac{dy}{dx} = 2y + x + 4 \qquad y(0) = 1$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) * h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

$$k_1 = f(x_i, y_i) \Rightarrow k_1 = 2 \times 1 + 0 + 4 = 6$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h) \Rightarrow f(0.1, 1.6) = 2 \times 1.6 + 0.1 + 4 = 7.3$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h) \Rightarrow f(0.1, 1.73) = 7.56$$

$$k_4 = f(x_i + h, y_i + k_3 h) \Rightarrow f(0.2, 2.512) = 9.224$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \times h = 1 + \frac{1}{6}(6 + 2 \times 7.3 + 2 \times 7.56 + 9.224) \times 0.2 = 2.4981$$

```
>> h=0.2;
>> Y1=1;
>> X1=0;
>> K1=2*Y1+X1+4
K1 =
     6
>> K2=2*(Y1+0.5*K1*h)+(X1+0.5*h)+4
K2 =
    7.3000
>> K3=2*(Y1+0.5*K2*h)+(X1+0.5*h)+4
K3 =
    7.5600
>> K4=2*(Y1+K3*h)+(X1+h)+4
K4 =
    9.2240
>> Y2=Y1+(1/6)*(K1+2*K2+2*K3+K4)*h
Y2 =
    2.4981
```

Second step X=X+h=0+0.2=0.2

```
>> Y1=2.4981;
>> X1=0.2;
>> K1=2*Y1+X1+4
K1 =
    9.1962
>> K2=2*(Y1+0.5*K1*h)+(X1+0.5*h)+4
K2 =
   11.1354
>> K3=2*(Y1+0.5*K2*h)+(X1+0.5*h)+4
K3 =
   11.5233
>> K4=2*(Y1+K3*h)+(X1+h)+4
K4 =
   14.0055
>> Y2=Y1+(1/6)*(K1+2*K2+2*K3+K4)*h
Y2 =
    4.7821
```

Third step X=X+h=0.2+0.2=0.4

```
>> Y1=4.7821;
>> X1=0.4;
>> K1=2*Y1+X1+4
K1 =
   13.9642
>> K2=2*(Y1+0.5*K1*h)+(X1+0.5*h)+4
K2 =
   16.8570
>> K3=2*(Y1+0.5*K2*h)+(X1+0.5*h)+4
K3 =
   17.4356
>> K4=2*(Y1+K3*h)+(X1+h)+4
K4 =
   21.1384
>> Y2=Y1+(1/6)*(K1+2*K2+2*K3+K4)*h
Y2 =
    8.2384
```