MTH231 DIFFERENTIAL EQUATIONS

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References

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1. INTRODUCTION TO DIFFERENTIAL EQUATIONS

What is a course in differential equations about?

You are familiar with algebra problems and solving algebraic equations. For example, the solutions to the quadratic equation

$$x^2 - x = 0$$

are easily found to be $x = 0$ and $x = 1$, which are **numbers.**

A differential equation **(DE)** is another type of equation where the unknown is not a number, but a **function**.

We will call this unknown $u(t)$ and think of it as a function of **time.**

- A DE also contains derivatives of the unknown function, which are also not known.
- So a DE is an equation that relates an unknown function to some of its derivatives.

1.1. DEFINITIONS AND TERMINOLOGY

Definition 1:

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be **a differential equation (DE).**

A simple example of a DE is

$$u'(t) = u(t).$$

where $u'(t)$ denotes the derivative of $u(t)$.

We want to find the solution of this equation. The question is this:

What function u(t) has a derivative that is equal to itself ?

The answer is $u(t) = e^t$, the exponential function.

We say this function is a solution of the DE, or it solves the DE.

But now we have another question:

Is it the only one?

If you try $u(t) = Ce^t$, where C is any constant whatsoever, you will also find it is a solution. **So differential equations have lots of solutions.**

This DE was very simple and we could guess the answer from our calculus knowledge. But, unfortunately (or, fortunately!), differential equations are usually more complicated.

Consider, for example, the DE

$$u''(t) + 2u'(t) + 2u(t) = 0.$$

This equation involves the unknown function and both its first and second derivatives.

We seek a function for which its second derivative, plus twice its first derivative, plus twice the function itself, is zero.

Now can you quickly guess a function $u(t)$ that solves this equation?

An answer is

$$u(t) = e^{-t}\cos t.$$

And

$$u(t) = e^{-t} sin\, t$$

works as well.

Let’s check this last one by using the product rule and calculating its derivatives:

$$
\begin{aligned}
u(t) &= e^{-t}\sin t, \\
u'(t) &= e^{-t}\cos t - e^{-t}\sin t, \\
u''(t) &= -e^{-t}\sin t - 2e^{-t}\cos t + e^{-t}\sin t.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& u''(t) + 2u'(t) + 2u(t) \\
= \; & -e^{-t}\sin t - 2e^{-t}\cos t + e^{-t}\sin t + 2(e^{-t}\cos t - e^{-t}\sin t) + 2e^{-t}\sin t \\
= \; & 0.
\end{aligned}
$$

So it works!

The function $u(t) = e^{-t} si\,n\,t$ solves the equation $u''(t) + 2u'(t) + 2u(t) = 0$.

In fact,

$$u(t) = Ae^{-t}\sin t + Be^{-t}\cos t$$

is a solution regardless of the values of the constants A and B. So, again, differential equations have lots of solutions.

Partly, the subject of differential equations is about developing methods for finding solutions.

Why differential equations? **Why are they so important to deserve a course of study?**

Well, differential equations arise naturally as models in areas of science, engineering, economics, and lots of other subjects.

Physical systems, biological systems, economic systems—all these are marked by **change**.

Differential equations model real-world systems by describing how they change.

The unknown function $u(t)$ could be

- the current in an electrical circuit,
- the concentration of a chemical undergoing reaction,
- the population of an animal species in an ecosystem,
- the demand for a commodity in a micro-economy.

Differential equations are laws that dictate change, and the unknown $u(t)$, for which we solve, describes **exactly how the changes occur.**

In this lesson, we study differential equations and their applications. We address two principal questions.

(1) How do we find an appropriate DE to model a physical problem?

(2) How do we understand or solve the DE after we obtain it?

We learn modeling by examining models that others have studied (such as Newton's second law), and we try to create some of our own through exercises.

We gain understanding and learn solution techniques by practice.

1. Differential Equations and Models

In science, engineering, economics, and in most areas where there is a quantitative component, we are greatly interested in describing how systems evolve in time, that is, in describing a system's **dynamics**.

In the simplest one dimensional case the state of a system at any time t is denoted by a function, which we generically write as $u = u(t)$.

We think of the **dependent variable** u as the state variable of a system that is varying with time t, which is the **independent variable**.

Thus, knowing u is tantamount to knowing what state the system is in at time t.

For example, $u(t)$ could be

- the population of an animal species in an ecosystem,
- the concentration of a chemical substance in the blood,
- the number of infected individuals in a flu epidemic,
- the current in an electrical circuit,
- the speed of a spacecraft,
- the mass of a decaying isotope,
- the monthly sales of an advertised item.

Knowledge of $u(t)$ for a given system tells us **exactly** how the state of the system is **changing in time**.

Figure 1.1 shows a time series plot of a generic state function.

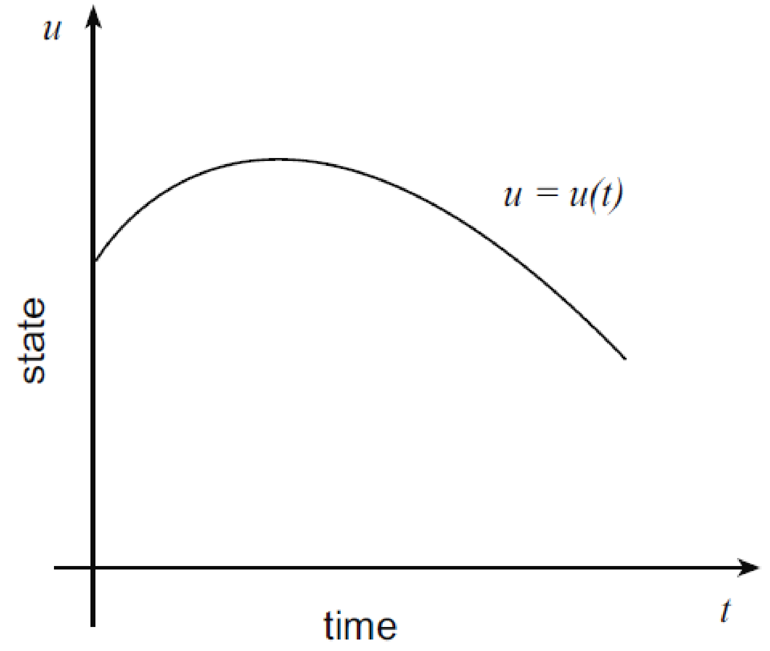

Figure 1.1 Time series plot of a generic state function $u = u(t)$ for a system.

We always use the variables y or u for a generic state; but if the state is “population”, then we may use p or N; if the state is voltage, we may use V . For mechanical systems we often use $x = x(t)$ for the position.

In summary, a differential equation is an equation that describes how a state $u(t)$ changes.

A common strategy in science, engineering, economics, etc., is to formulate a basic principle in terms of a differential equation for an unknown state that characterizes a system and then solve the equation to determine the state, thereby determining how the system evolves in time.

Classification of Differential Equations

CLASSIFICATION BY TYPE

If an equation contains only ordinary derivatives of **one or more dependent variables** with respect to a **single independent variable** it is said to be **an ordinary differential equation (ODE).**

For example,

A DE can contain more
than one dependent variable

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \qquad \text{and} \qquad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

are ordinary differential equations.

An equation involving partial derivatives of **one or more dependent variables** of **two or more independent variables** is called **a partial differential equation (PDE)**. For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are partial differential equations.

Ordinary differential equations will be considered in this course. PDE is a subject of another lesson in itself.

Throughout this lesson ordinary derivatives will be written by using either the **Leibniz notation**

$$dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$$

or the **prime notation**

$$y', y'', y''', \ldots .$$

The equations in ODE examples can be written as

$$y' + 5y = e^x \text{ and } y'' - y' + 6y = 0.$$

Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative of y is written $\frac{d^n y}{dx^n}$ or $y^{(n)}$.

Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

$$\frac{d^2 \overset{\substack{\text{unknown function} \\ \text{or dependent variable}}}{\overbrace{x}}}{d\underbrace{t}_{\text{independent variable}}^2} + 16x = 0$$

it is immediately seen that the symbol x now represents a dependent variable, whereas the independent variable is t.

- You should also be aware that in physical sciences and engineering, Newton's **dot notation** is sometimes used to denote derivatives with respect to time t. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$.

- Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in PDE examples becomes

$$u_{xx} = u_{tt} - 2u_t.$$

CLASSIFICATION BY ORDER

The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\overset{\text{second order}}{\frac{d^2y}{dx^2}} + 5\left(\overset{\text{first order}}{\frac{dy}{dx}}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation.

First-order ordinary differential equations are occasionally written in differential form $M(x,y)dx + N(x,y)dy = 0$.
For example, if we assume that y denotes the dependent variable in $(y-x)dx + 4xdy = 0$, then $y' = \frac{dy}{dx}$, so by dividing by the differential dx, we get the alternative form $4xy' + y = x$.

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$\boxed{F(x,\ y,\ y',\ldots,y^{(n)}) = 0,} \qquad (1)$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$.

For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the above form uniquely for the highest derivative $y^{(n)}$ in terms of the remaining n+1 variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}),$$

where f is a real-valued continuous function, is referred to as the normal form of Eq.(1).

Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations. For example, **the normal form** of

- the first-order equation $4xy' + y = x$ is $y' = (x - y)/4x$.
- the normal form of the second-order equation $y'' - y' + 6y = 0$ is $y'' = y' - 6y$.

CLASSIFICATION BY LINEARITY

An nth-order ordinary differential equation (1) is said to be linear if F is linear in $y, y', \ldots, y^{(n)}$.This means that an nth-order ODE is linear when (1) is

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' - g(x) = 0$$

or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} = g(x) \qquad (2)$$

Two important special cases of (2) are linear first-order ($n = 1$) and linear second order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad \text{and} \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In the additive combination on the left-hand side of equation (1) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \cdots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \cdots, a_n$ of $y, y', y'', \cdots, y^{(n)}$ depend at most on the independent variable x.

The equations

$$(y - x)dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad \text{and} \quad \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, linear first-, second-, and third-order ordinary differential equations.

We have just demonstrated that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation. Therefore

$$\overset{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y\\ \downarrow}}{(1-y)}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \overset{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y\\ \downarrow}}{\sin y} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \overset{\substack{\text{nonlinear term:}\\ \text{power not 1}\\ \downarrow}}{y^2} = 0$$

are examples of nonlinear first-, second-, and fourth-order ordinary differential equations, respectively.

System of Differential Equations

A system of differential equations is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if $x \; and \; y$ denote dependent variables and t denotes the independent variable, then a system of two first order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y).$$