Lecture 1: Sytems of Linear Equations

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Systems of Linear Equations

Definition

A linear equation in n unknowns is an equation which has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b. \tag{1}$$

Here $x_1, x_2, ..., x_n$ are unknowns, $a_1, a_2, ..., a_n$, and b are real or complex constants.

A solution of a linear equation is a sequence of $s_1, s_2, ..., s_n$, such that (1) is satisfied when

$$x_1 = s_1, x_2 = s_2, ..., x_n = s_n.$$

Systems of Linear Equations

Examples

A solution of the linear equation

$$3x_1 + 2x_2 - 5x_3 = -7$$

is $x_1 = 2, x_2 = 1$, and $x_3 = 3$.
The equations

$$2\sqrt{x_1} - 5x_3 = 12$$

and

$$3x_1x_2 + 7x_3 = -1$$

are not linear equations because they are not in the form (1).

Systems of Linear Equations

Definition

A system of linear equations is a set of m linear equations in n unknowns which has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_1 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned} \tag{2}$$

Here a_{ij} are known constants.

A solution of the linear system (2) is a sequence $s_1, s_2, ..., s_n$, such that every equation in the system (2) is satisfied when $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$.

- If the linear system (2) has no solution, then it is called inconsistent.
- If the linear system (2) has one or more solutions, then it is called consistent.
- If $b_1 = b_2 = \cdots = b_n = 0$, then the linear system (2) is called a homogeneous system. Note that a solution to a homogeneous system such that $x_1 = x_2 = \cdots = x_n = 0$ is called a trivial solution, otherwise it is called a nontrivial solution.

Systems of Linear Equations

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1. Interchange the i-th and j-th equations

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2. Multiply an equation by a non zero constant
3. Add a multiple of i-th equation to the another j-th equation.

Example

Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

If we add (-2) times the first equation to the second one and (-1) times the first equation to the third one, we get

$$\begin{aligned} -3x_2 - 9x_3 &= -9 \\ -3x_2 - 2x_3 &= -2. \end{aligned}$$

Example

If we add (-1) times the first equation to the second one, we get $x_3 = 1$.
Then substituting the value of x_3 into the first equation of second linear system, we get $x_2 = 0$.
Finally substituting these values of x_2 and x_3 into the first equation of first linear system, we find that $x_1 = 2$.