Lecture 2: Matrices

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Matrices

Definition

An $m \times n$ matrix A is a rectangular array of mn scalars, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} .$$

- We say that the size of the matrix is $m \times n$.
- If $m = n$, A is called a square matrix.
- The scalar a_{ij} which is in the i-th row and j-th column of A is called the (i, j)-th entry of A.
- We write the matrix A as $A = [a_{ij}]_{m \times n}$.
- If all corresponding entries of $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, then they are called an equal matrix.

Matrix Operations

Matrix addition: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.
Scalar multiplication: Let $A = [a_{ij}]_{m \times n}$ and $r \in \mathbb{R}$. Then $rA = [ra_{ij}]_{m \times n}$.
Transpose: Let $A = [a_{ij}]_{m \times n}$. Then $A^T = [a_{ji}]_{n \times m}$.
Matrix multiplication: Let $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$. The product of A and B is defined by

$$AB = [c_{ij}]_{m \times n}, \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Note that the product of A and B is defined only when the number of columns of A is equal to the number of rows of B.

Properties of Matrix Operations

Theorem (Matrix addition)

Let A, B, and C be $m \times n$ matrices.

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) $A + 0 = A$ *(0 is $m \times n$ zero matrix)*

(d) $A + (-A) = 0$ *($-A$ is the negative of A)*

Properties of Matrix Operations

Theorem (Scalar multiplication)

Let A, B be $m \times n$ matrices and $r, s \in \mathbb{R}$.

(a) $r(sA) = (rs)A$
(b) $(r+s)A = rA + sA$
(c) $r(A+B) = rA + rB$
(d) $A(rB) = r(AB) = (rA)B$.

Properties of Matrix Operations

Theorem (Matrix multiplication)

Let A, B, and C are matrices of the appropriate sizes.
(a) $A(BC) = (AB)C$
(b) $(A+B)C = AC + BC$
(c) $C(A+B) = CA + CB$.

Remark: Note that AB need not always equal BA!
For example, let $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ while $BA = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Properties of Matrix Operations

Theorem (Transpose)

Let A, B are matrices of the appropriate sizes and $r \in \mathbb{R}$.

(a) $\left(A^T\right)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(AB)^T = B^T A^T$

(d) $(rA)^T = rA^T$.

Non singular Matrix

Definition

Identity matrix: The matrix $I_n = [d_{ij}]_{n \times n}$ is called the $n \times n$ identity matrix whose entries satisfy the following rule:

$$d_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Nonsingular matrix: The matrix $A = [a_{ij}]_{n \times n}$is called the nonsingular matrix if there exists a matrix $B = [b_{ij}]_{n \times n}$ such that $AB = BA = I_n$. The matrix B is called an inverse of A.

Theorem

The inverse of a matrix is unique, if it exists.

Nonsingular Matrix

Theorem (Properties of inverse of a matrix)

Let A and B be $n \times n$ matrices.

(a) $\left(A^{-1}\right)^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$.