Lecture 4: Determinants

Elif Tan

Ankara University

Determinants

Definition (Permutation)

Let $S = \{1, 2, ..., n\}$ be a set of integers, arranged in ascending order. A rearrangement $j_1 j_2 ... j_n$ of the elements of S is called a permutation of S.

The set of all permutations of $S =: S_n$
The number permutations of $S_n = n!$

Definition (Inversion)

A permutation $j_1 j_2 ... j_n$ is aid to have an inversion if a larger integer, j_r, precedes a smaller one, j_s.

If the total number of inversions is even$\Rightarrow$a permutation is called even.
If the total number of inversions is odd$\Rightarrow$a permutation is called odd.
There exists $\frac{n!}{2}$ even and $\frac{n!}{2}$ odd permutations in S_n.

Determinants

Definition (Determinant)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant function is defined by

$$\det(A) = \sum_{j_1 j_2 \ldots j_n \in S_n} (\pm)\, a_{1j_1} a_{2j_2} \ldots a_{nj_n}.$$

(If the permutation $j_1 j_2 \ldots j_n$ is even, then the sign is taken $+$, otherwise $-$)

Example

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $S_2 = \{12, 21\}$.

$$\det(A) = \sum_{j_1 j_2 \in S_2} (\pm)\, a_{1j_1} a_{2j_2} = a_{11}a_{22} - a_{12}a_{21}.$$

Properties of determinants

Theorem

Let A be an $n \times n$ matrix.

1. $\det\left(A_{r_i \leftrightarrow r_j}\right) = -\det(A)$
2. $\det\left(A_{kr_i \to r_i}\right) = k \det(A)$
3. $\det\left(A_{kr_i + r_j \to r_j}\right) = \det(A)$
4. $\det(A) = \det\left(A^T\right)$
5. *If two rows(columns) of A are equal* $\Rightarrow \det(A) = 0$
6. *If A consists a zero row(column)* $\Rightarrow \det(A) = 0$
7. *If A is upper(lower) triangular matrix* $\Rightarrow \det(A) = a_{11}a_{22}...a_{nn}$ $(\det(I_n) = 1)$
8. $\det(AB) = \det(A)\det(B)$

Properties of determinants

Theorem

If A is an $n \times n$ nonsingular matrix $\Leftrightarrow \det(A) \neq 0$.

Corollary

1. $Ax = b$ *has a unique solution* $\Leftrightarrow \det(A) \neq 0$.
2. $Ax = 0$ *has a nontrivial solution* $\Leftrightarrow \det(A) = 0$.
3. *If A is $n \times n$ nonsingular matrix* $\Leftrightarrow \det\left(A^{-1}\right) = \frac{1}{\det(A)}$.

Cofactor Expansion

Now we show an another method to evaluate the determinant of an $n \times n$ matrix A.

Definition (Minor)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The minor of a_{ij} is defined as $\det(M_{ij})$, where M_{ij} is $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and j-th column of A.

Definition (Cofactor)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor of a_{ij} is defined as $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

Cofactor Expansion

Theorem

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij} \quad \textit{(expansion of } \det(A) \textit{ along the } i\textit{-th row)}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} A_{ij} \quad \textit{(expansion of } \det(A) \textit{ along the } j\textit{-th column)}$$

Remark: It is useful to expand along the row (column) which has more zero.

Cofactor Expansion

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. If we expand $\det(A)$ along the $1st$ row, we get

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= 1(-1)^{1+1}\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \\ &\quad +3(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= -21. \end{aligned}$$

Applications of determinants: Finding inverse of a matrix

Now we show another method for finding inverse of a nonsingular matrix.

Definition (Adjoint)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The adjoint of A is defined as

$$adj\,(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ji} is the cofactor of a_{ji}.

Theorem

If $A = [a_{ij}]$ *be an* $n \times n$ *matrix and* $\det(A) \neq 0$, *then*
$A^{-1} = \frac{1}{\det(A)} adj\,(A)$.

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. Find A^{-1}, if it exists.

Solution: Since $\det(A) = -21 \neq 0$, the matrix A has an inverse. If we evaluate the matrix $adj(A)$, we have

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} adj(A) \\ &= \frac{-1}{21} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{5}{21} & \frac{9}{21} \\ \frac{5}{21} & \frac{2}{21} & -\frac{9}{21} \\ \frac{3}{21} & -\frac{3}{21} & \frac{3}{21} \end{bmatrix}. \end{aligned}$$

Applications of determinants: Cramer's rule

Theorem

Consider the linear system of n equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

Let A be the coefficient matrix of given linear system. If $\det(A) \neq 0$, *then the linear system has a unique solution as*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)},$$

where A_i is the matrix otained from A by replacing i-th column of A by b.

Applications of determinants: Cramer's rule

Example

Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

Since $\det(A) = -21 \neq 0$, the linear system has unique solution, and the solutions can be obtained by using Cramer's rule as:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 5 & 2 & 3 \\ 1 & 1 & -3 \\ 3 & -1 & 1 \end{vmatrix}}{-21} = 2,$$

Applications of determinants: Cramer's rule

Example

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & -3 \\ 1 & 3 & 1 \end{vmatrix}}{-21} = 0,$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix}}{-21} = 1.$$