Lecture 13: Gram-Schmidt Process

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Definition (Orthonormal Basis)

Let V be an inner product space and $S = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for the vector space V. S is called an orthonormal basis if the followings are satisfied:
$(i)\, \forall i \neq j, \langle u_i, u_j \rangle = 0$
$(ii)\, i = 1, ..., n, \langle u_i, u_i \rangle = 1.$

To obtain this orthonormal basis we use a method which is called Gram-Schmidt process.

Gram-Schmidt Process

Theorem

Let $T = \{w_1, w_2, \ldots, w_n\}$ *be an orthonormal basis for the inner product space* V *and* $v \in V$*. Then*

$$v = c_1 \odot w_1 \oplus c_2 \odot w_2 \oplus \cdots \oplus c_n \odot w_n$$

where

$$c_i = \langle v, w_i \rangle .$$

We determine the coordinates of the vector by using inner product instead of solving linear system!

Gram-Schmidt Process

Theorem (Gram-Schmidt Process)

Let V be an inner product space and $\{0\} \neq W < V$ and $\dim W = m$. Then there exists an orthonormal basis $T = \{w_1, w_2, \ldots, w_n\}$ for W.

Gram-Schmidt Process

Sketch of the proof: Let $S = \{u_1, u_2, \ldots, u_m\}$ be any basis for W.

- $v_1 := u_1$
- $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \odot v_1$ (Note that $v_1 \perp v_2$)
- $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} \odot v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \odot v_2$ (Note that $v_1 \perp v_2$, $v_1 \perp v_3$, $v_2 \perp v_3$)
- Similarly, we obtain m-orthogonal vectors as

$$v_m = u_m - \odot v_1 - \frac{\langle u_m, v_2 \rangle}{\langle v_2, v_2 \rangle} \odot v_2 - \cdots - \frac{\langle u_m, v_{m-1} \rangle}{\langle v_{m-1}, v_{m-1} \rangle} \odot v_{m-1}.$$

- Thus, we obtain an orthogonal set $K = \{v_1, v_2, \ldots, v_m\}$. Since K is a linear independent set in m-dimensional vector space, K is a basis for W.

Gram-Schmidt Process

Now, if we let

$$w_i = \frac{v_i}{\|v_i\|}$$

for $i = 1, 2, \ldots, m$, then $T = \{w_1, w_2, \ldots, w_m\}$ is an orthonormal basis for W.

Gram-Schmidt Process

The matrix of the inner product with respect to the ordered orthogonal basis K is

$$B = \begin{bmatrix} \langle v_1, v_1 \rangle & 0 & \cdots & 0 \\ 0 & \langle v_2, v_2 \rangle & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \langle v_n, v_n \rangle \end{bmatrix}.$$

Gram-Schmidt Process

The matrix of the inner product with respect to the ordered orthonormal basis T is

$$I_m = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Gram-Schmidt Process

For $u, v \in V$, we have $[u]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ and $[v]_T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

If we use the orthonormal basis T, then

$$\begin{aligned} \langle u, v \rangle &= \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} I_m \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_m b_m \end{aligned}$$

which is the standard inner product.

References

1) B. Kolman and D.R. Hill. Elemantery Linear Algebra with Applications. Pearson I.E. (9th Edition)

2) L.Spence,A. Insel, S. Friedberg. Elemantery Linear Algebra A Matrix Aproach. Pearson I.E. (2nd Edition)