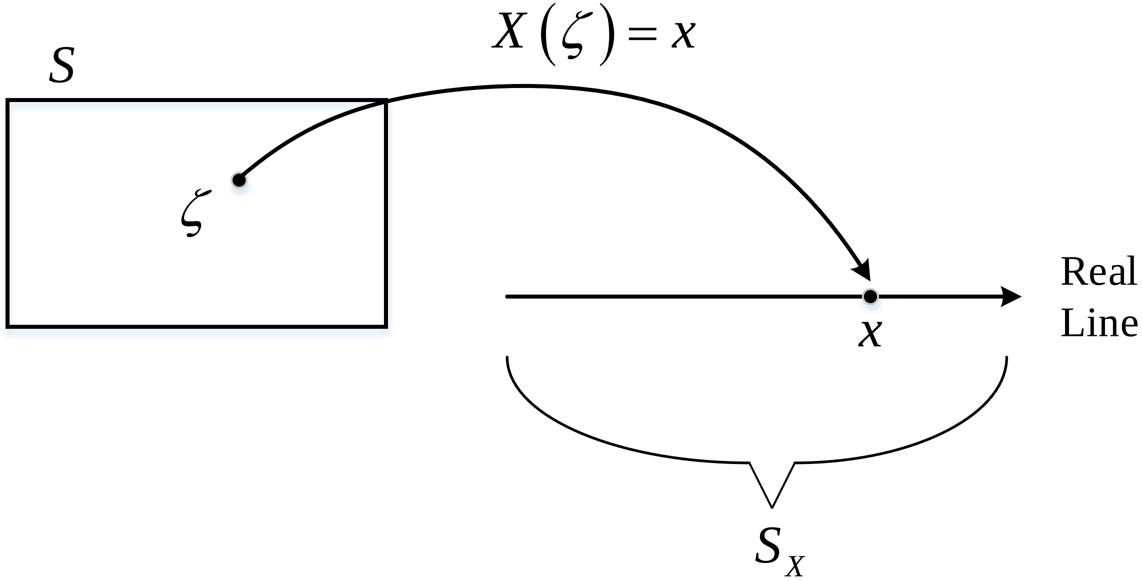
DISCRETE RANDOM VARIABLES

BASIC CONCEPTS

A **random variable:** assigns a numerical value to the outcome of the experiment (a function)



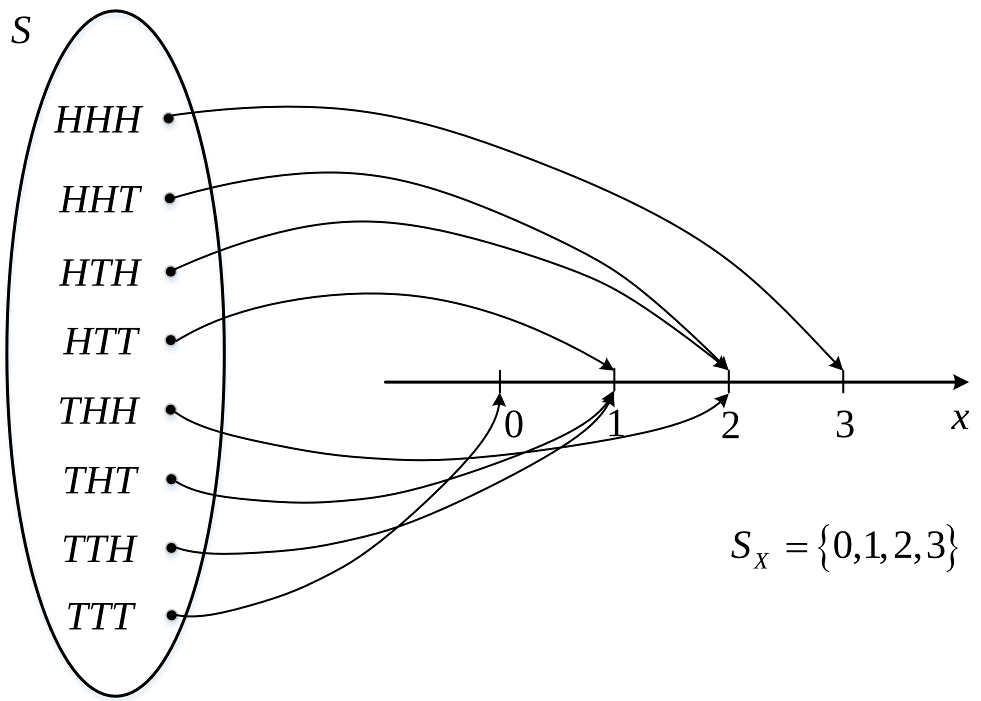
Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

BASIC CONCEPTS

Example: tossing coin three times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

X: <u>the number of heads.</u>



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BASIC CONCEPTS

Concepts Related to Discrete Random Variables

- A **discrete random variable** can take a finite or countably infinite number of values.
- **Probability mass function (PMF):** probability of each numerical value
- A function of a random variable defines a new random variable
    - PMF of the new random variable can be obtained from the PMF of the original random variable.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

PROBABILITY MASS FUNCTIONS

Characterizing a random variable : Probability Mass Function (PMF)

$$p_X(x) = P(X = x)$$

Notation:

X : random variable

x : any possible value of random variable X

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

PROBABILITY MASS FUNCTIONS

Example (textbook p 74-75):

Experiment: two independent fair coin tosses
X : the number of heads obtained.

The PMF of X is

$$p_X(x) = \begin{cases} 1/4, & \text{if } x = 0 \text{ or } x = 2, \\ 1/2, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\sum_x p_X(x) = 1$$ (<u>additivity</u> and <u>normalization</u> axioms)

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PROBABILITY MASS FUNCTIONS

Calculation of the PMF of a Random Variable *X*:

For each possible value *x* of *X*

1) Find all possible outcomes

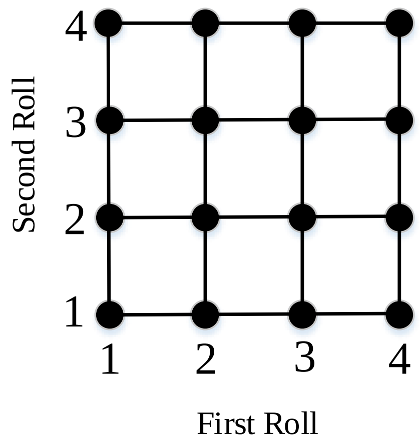
2) Sum their probabilities

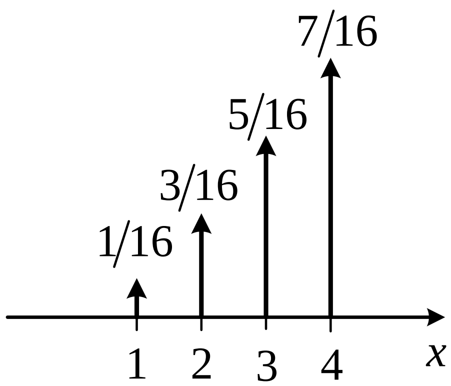
PROBABILITY MASS FUNCTIONS

Example (textbook p. 77)

Experiment: two rolls of a four-sided fair die. X : the maximum of two rolls. Let's find the PMF of X.

Sample Space Pair of Rolls





$$p_X(1) = P\left[\{(1,1)\}\right] = 1/16$$

$$p_X(2) = P\left[\{(2,1),(1,2),(2,2)\}\right] = 3/16$$

$$p_X(3) = P\left[\{(3,1),(1,3),(3,2),(2,3),(3,3)\}\right] = 5/16$$

$$p_X(4) = P\left[\{(4,1),(1,4),(4,2),(2,4),(4,3),(3,4),(4,4)\}\right] = 7/16$$

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PROBABILITY MASS FUNCTIONS
The Discrete Uniform Random Variable

$$p_X(k) = P(X = k) = \frac{1}{N}, \quad \text{for } k \in \{j+1, \ldots, j+N\}$$

The Bernoulli Random Variable

Consider the toss of a <u>biased coin</u>,
The probability of head: p,
The probability of tail: $1-p$.

$$X = \begin{cases} 1, & \text{if a head,} \\ 0, & \text{if a tail.} \end{cases}$$

Its PMF

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0. \end{cases}$$

$p_X(1) + p_X(0) = 1 \rightarrow$ Normalization Property

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PROBABILITY MASS FUNCTIONS

The Binomial Random Variable

Consider tossing a biased coin n times.
At each toss:
The probability of head: p, (independent of prior tosses)
The probability of tail: $1-p$.

X : the number of heads in n-toss sequence.

The PMF of X (by using binomial probabilities)

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2 \ldots, n$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 \qquad \rightarrow \quad \text{Normalization Property}$$

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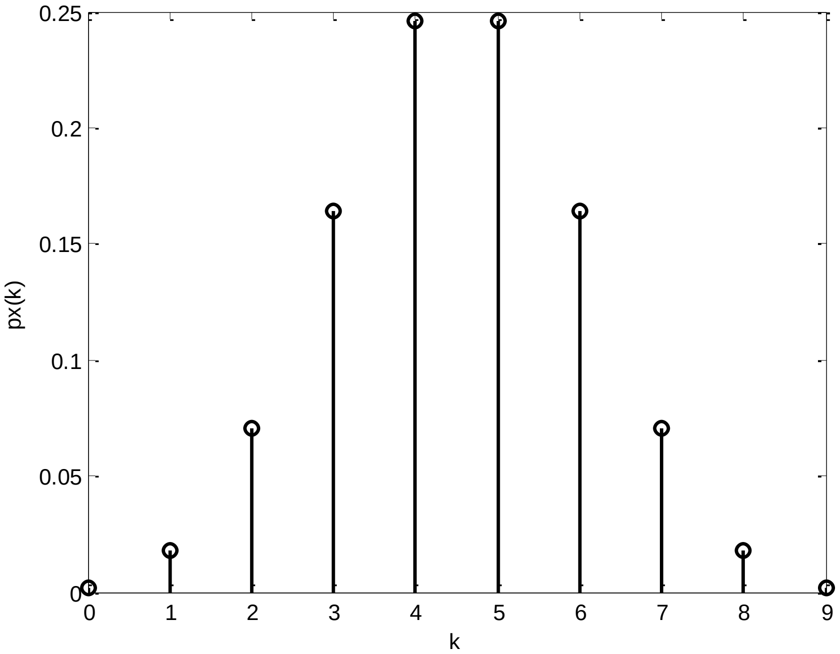
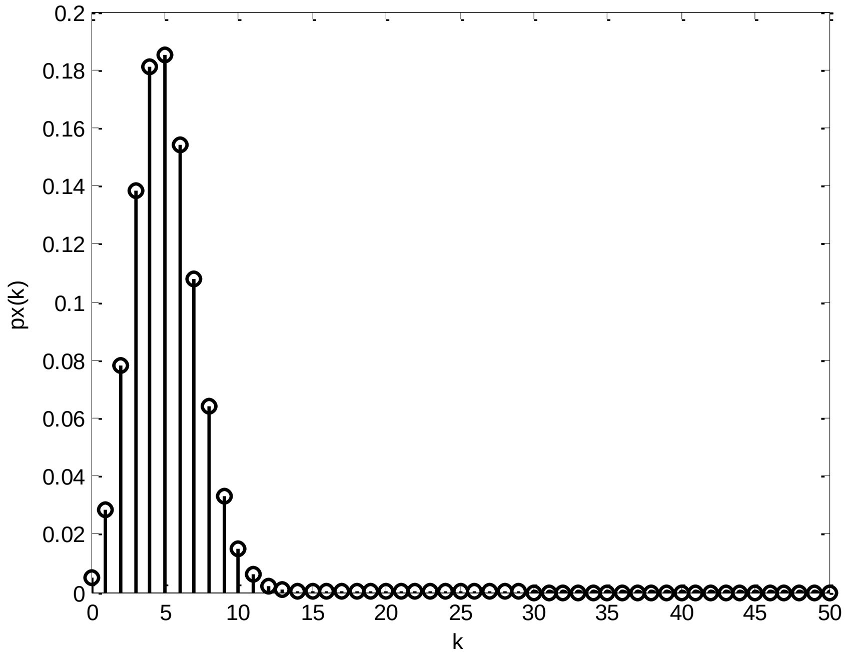
PROBABILITY MASS FUNCTIONS

The Binomial Random Variable



$n = 9, \quad p = 0.5$



$n = 50, \quad p = 0.1$

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PROBABILITY MASS FUNCTIONS
The Geometric Random Variable

The number of independent trials needs to obtain the first "success".

Consider the experiment where a coin is tossed repeatedly and independently.
probability of head: p.

The **geometric** random variable: the number X of tosses needed for a head for the first time.
$k - 1$ tails followed by a head, the probability $(1-p)^{k-1}p$,
The PMF of X:

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

This is a legitimate PMF because

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p\sum_{k=0}^{\infty}(1-p)^k = p \times \frac{1}{1-(1-p)} = 1$$

Note that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$, for $|x| < 1$ (Geometric Series)

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PROBABILITY MASS FUNCTIONS
The Poisson Random Variable

- Number of demands for telephone connections
- Number of traffic accidents in a particular street in a day

PMF :

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

where lambda is a positive parameter. This is a legitimate PMF since

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right) = e^{-\lambda} \times e^{\lambda} = 1$$

Note that the Taylor series for the exponential function $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

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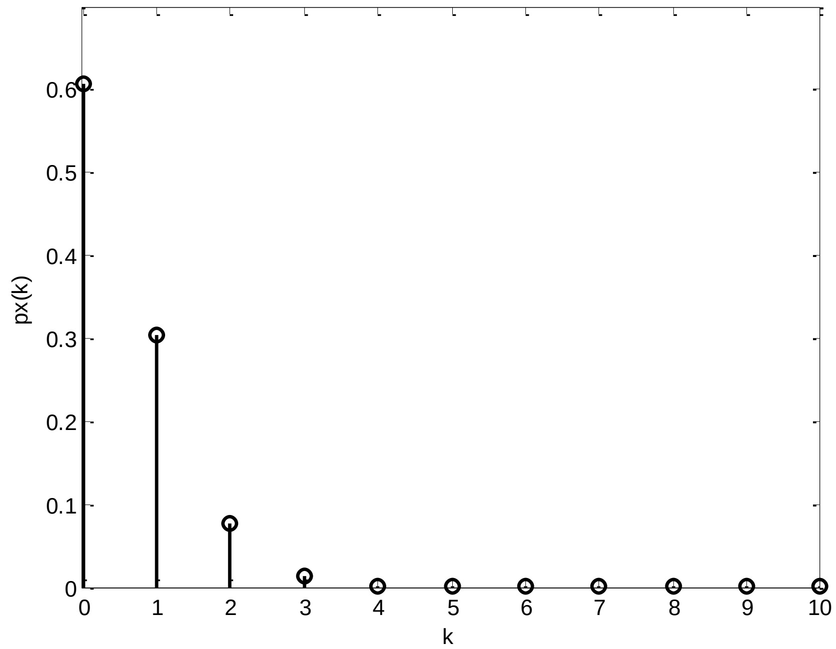
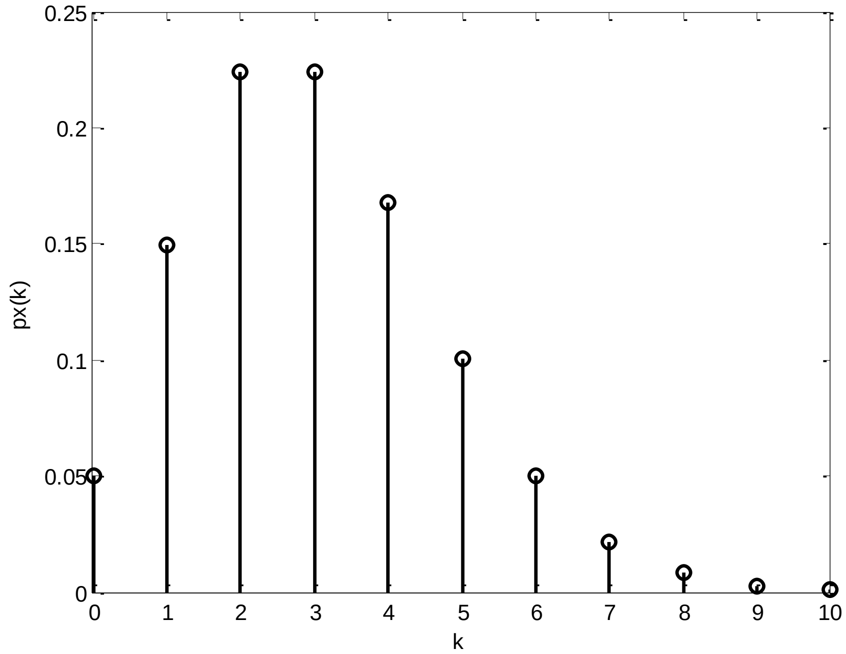
PROBABILITY MASS FUNCTIONS

The Poisson Random Variable



$\lambda = 0.5$



$\lambda = 3$

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