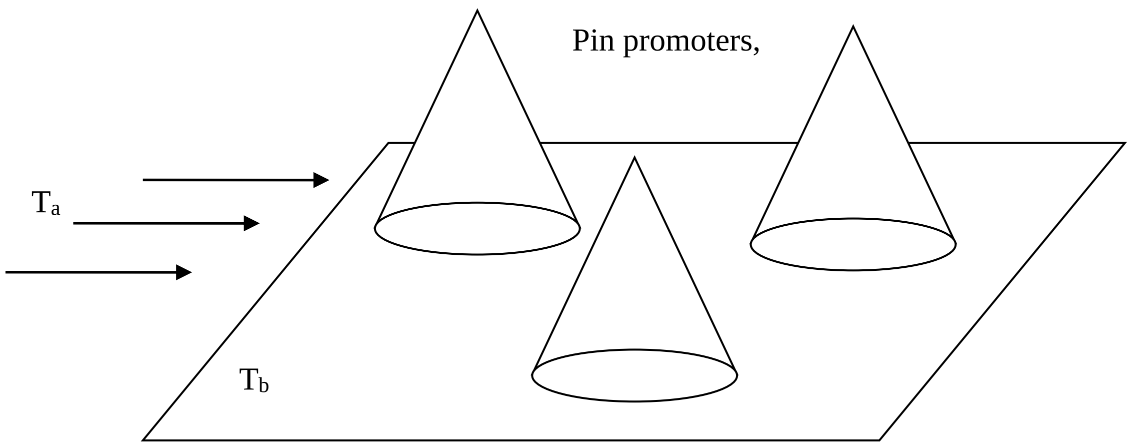
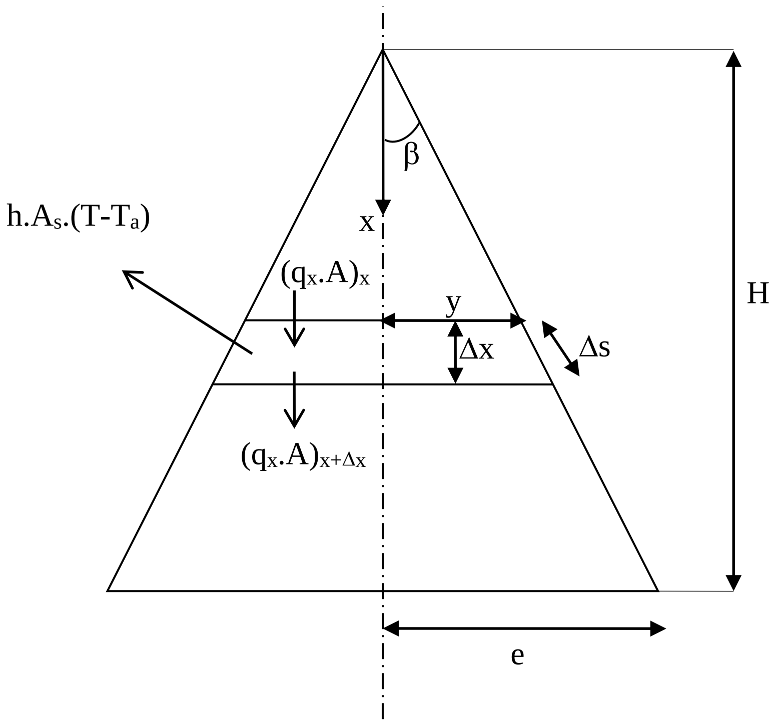
Solution of the second order linear differential equation with variable coefficient by Series (Bessel's equation, Modified Bessel's Equation) (R. G. Rice, D. D. Do, Applied Mathematics And Modeling For Chemical Engineers, John Wiley and Sons, 995)

Example :

Pin promoters attached to the plate shown in figure are used to enhance heat transfer. Find an expression to compute the temperature profile within a pin promoter, assuming temperature varies mainly in the x direction. The temperature of the plate T_b, fluid temperature T_a, and heat transfer coefficient h are constant. (Solve the differential equation by Generalized Bessel's equation)





$$A = \pi x^2 \left(\frac{e}{H}\right)^2 \text{ and } A_s = 2\pi y \Delta s$$

Note that; $y = \frac{e}{H} x$ and $\Delta s = \frac{\Delta x}{\cos\beta}$

Solution:

$$q_x.A\big|_x - q_x.A\big|_{x+\Delta x} - hA_s\left(T - T_a\right) = 0$$

$$A = \pi x^2\left(\frac{e}{H}\right)^2, \quad A_s = 2\pi y\Delta s$$

$$y = \frac{e}{H}x \text{ and } \Delta s = \frac{\Delta x}{\cos\beta} \qquad A_s = 2\pi\frac{e}{H}x\frac{\Delta x}{\cos\beta}$$

$$q_x.\pi x^2\left(\frac{e}{H}\right)^2\Bigg|_x - q_x.\pi x^2\left(\frac{e}{H}\right)^2\Bigg|_{x+\Delta x} - h2\pi\frac{e}{H}x\frac{\Delta x}{\cos\beta}\left(T - T_a\right) = 0$$

divide by $\pi\left(\frac{e}{H}\right)^2\Delta x$

$$\frac{q_x.x^2\big|_x - q_x.x^2\big|_{x+\Delta x}}{\Delta x} - \frac{2hx}{\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0 \qquad \text{limit } \Delta x \to 0$$

$$-\frac{d}{dx}\left(q_x.x^2\right) - \frac{2hx}{\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0 \qquad \text{Fourier's law: } q_x = -k\frac{dT}{dx}$$

$$-\frac{d}{dx}\left(-k\frac{dT}{dx}.x^2\right) - \frac{2hx}{\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0$$

$$k\left[\frac{d}{dx}\left(\frac{dT}{dx}.x^2\right)\right] - \frac{2hx}{\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0$$

$$\frac{d}{dx}\left(\frac{dT}{dx}.x^2\right) - \frac{2hx}{k\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0$$

$$\frac{d^2T}{dx^2}x^2 + \frac{dx^2}{dx}\frac{dT}{dx} - \frac{2hx}{k\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0$$

$$x^2\frac{d^2T}{dx^2} + 2x\frac{dT}{dx} - \frac{2hx}{k\frac{e}{H}\cos\beta}\left(T - T_a\right) = 0$$

BC1: $x = 0 \qquad \frac{dT}{dx}\Big|_{x=0} = 0$

BC2: $x = H \qquad T = T_b$

In order to apply Bessel's eq., let $y = (T-T_a)$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-\frac{2hx}{k\frac{e}{H}\cos\beta}y=0$$

$$\text{BC1: } x=0 \quad \left.\frac{dy}{dx}\right|_{x=0}=0$$

$$\text{BC2: } x=H \quad y=T_b-T_a$$

Generalized Bessel's eq.

$$x^2\frac{d^2y}{dx^2}+x\left(a+2bx^r\right)\frac{dy}{dx}+\left[c+dx^{2s}-b(1-a-r)x^r+b^2x^{2r}\right]y=0$$

$$\left(a+2bx^r\right)=2 \qquad a=2;b=0$$

$$\left[c+dx^{2s}-b(1-a-r)x^r+b^2x^{2r}\right]=-\frac{2hx}{k\frac{e}{H}\cos\beta}$$

$$\left[c+dx^{2s}\right]=-\frac{2hx}{k\frac{e}{H}\cos\beta} \qquad c=0;d=-\frac{2h}{k\frac{e}{H}\cos\beta};s=1/2$$

Order of eq. $\quad k=\frac{1}{s}\sqrt{\left(\frac{1-a}{2}\right)^2-c}=\frac{1}{1/2}\sqrt{\left(\frac{1-2}{2}\right)^2}=1$

$\sqrt{d}/s$ is imaginary and k=1 then, $Z_k=I_k$ and $Z_{-k}=K_k$

$$y=x^{\left(\frac{1-a}{2}\right)}e^{\frac{-bx^r}{r}}\left[c_1Z_k\left(\frac{\sqrt{|d|}}{s}x^s\right)+c_2Z_{-k}\left(\frac{\sqrt{|d|}}{s}x^s\right)\right]$$

$$y=x^{\left(\frac{-1}{2}\right)}\left[c_1I_1\left(\frac{\sqrt{\frac{2h}{k\frac{e}{H}\cos\beta}}}{1/2}x^{1/2}\right)+c_2K_1\left(\frac{\sqrt{\frac{2h}{k\frac{e}{H}\cos\beta}}}{1/2}x^{1/2}\right)\right]$$

$$y=\frac{1}{\sqrt{x}}\left[c_1I_1\left(2\sqrt{\frac{2h}{k\frac{e}{H}\cos\beta}}\sqrt{x}\right)+c_2K_1\left(2\sqrt{\frac{2h}{k\frac{e}{H}\cos\beta}}\sqrt{x}\right)\right]$$