Physics 122: Electricity & Magnetism – Lecture 1

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Vector Basics

- We will be using vectors a lot in this course.
- Remember that vectors have both magnitude and direction e.g. $a, \ \theta$
- You should know how to find the components of a vector from its magnitude and direction

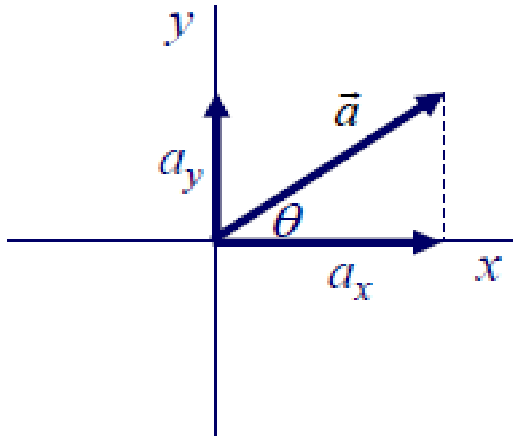
$$a_x = a\cos\theta$$
$$a_y = a\sin\theta$$

- You should know how to find a vector's magnitude and direction from its components

$$a = \sqrt{a_x^2 + a_y^2}$$
$$\theta = \tan^{-1} a_y / a_x$$

Ways of writing vector notation

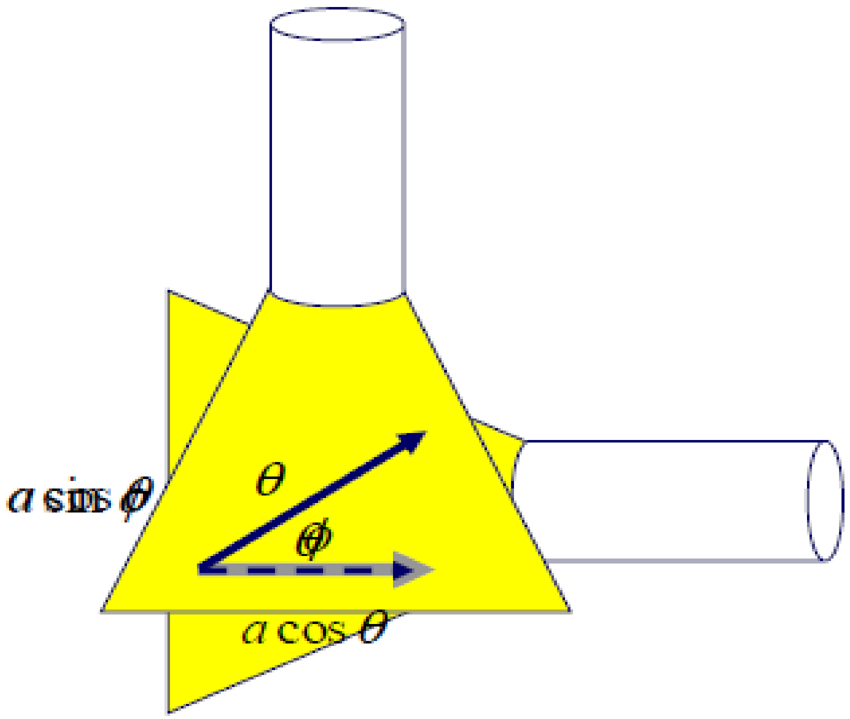
$$\mathbf{F} = m\mathbf{a}$$
$$\vec{F} = m\vec{a}$$
$$\underline{F} = m\underline{a}$$



Projection of a Vector and Vector Components

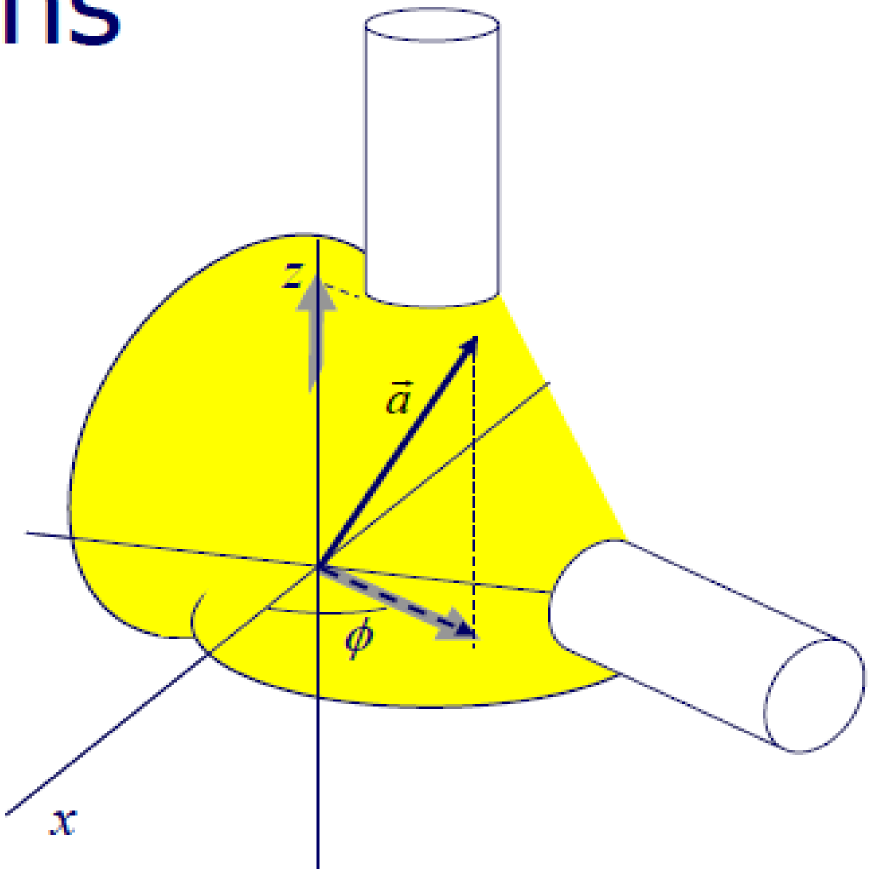
- When we want a component of a vector along a particular direction, it is useful to think of it as a projection.
- The projection always has length $a\cos\theta$, where a is the length of the vector and θ is the angle between the vector and the direction along which you want the component.
- You should know how to write a vector in unit vector notation

$$\vec{a} = a_x\hat{i} + a_y\hat{j} \quad \text{or} \quad \vec{a} = (a_x, a_y)$$



Projection of a Vector in Three Dimensions

- Any vector in three dimensions can be projected onto the x-y plane.
- The vector projection then makes an angle ϕ from the x axis.
- Now project the vector onto the z axis, along the direction of the earlier projection.
- The original vector a makes an angle θ from the z axis.



Vector Basics

- You should know how to generalize the case of a 2-d vector to three dimensions, e.g. 1 magnitude and 2 directions $a,\ \theta,\ \phi$
- Conversion to x, y, z components

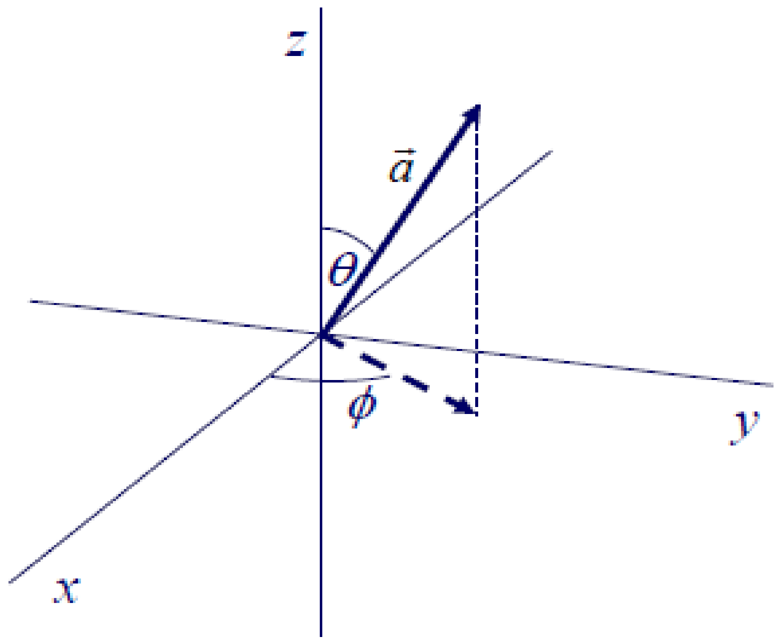
$$a_x = a\sin\theta\cos\phi$$
$$a_y = a\sin\theta\sin\phi$$
$$a_z = a\cos\theta$$

- Conversion from x, y, z components

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$
$$\theta = \cos^{-1} a_z / a$$
$$\phi = \tan^{-1} a_y / a_x$$

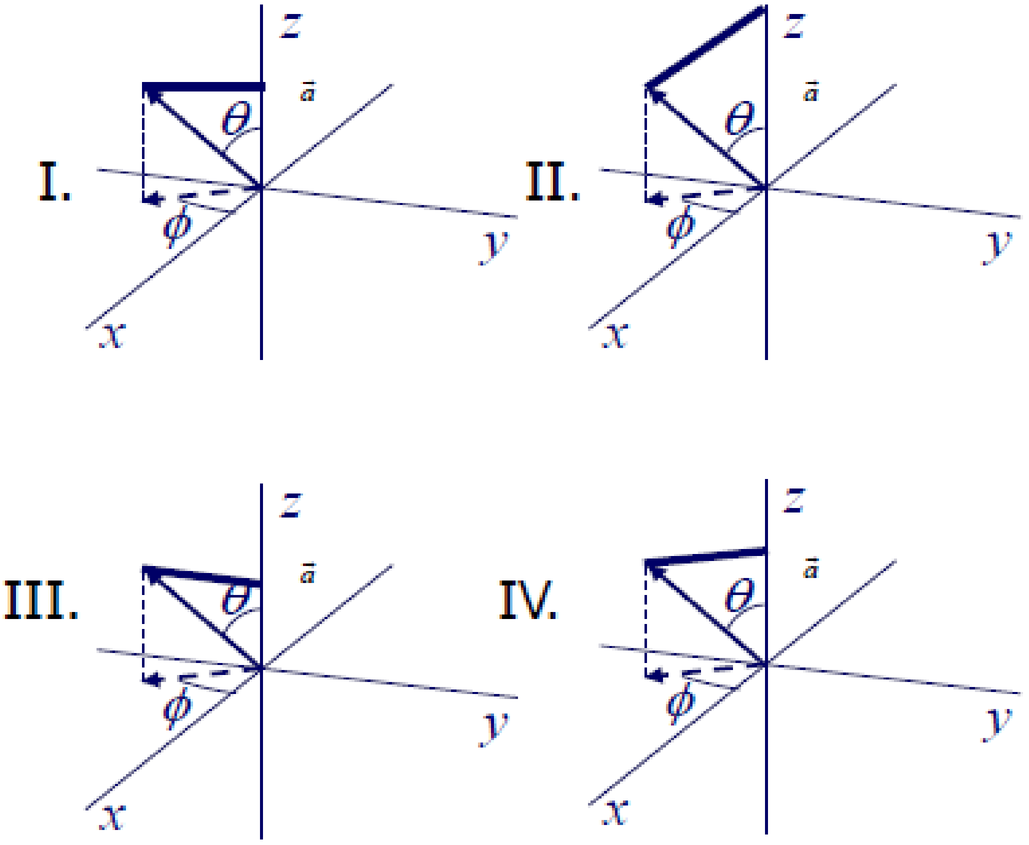
- Unit vector notation:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$



Seeing in 3 Dimensions

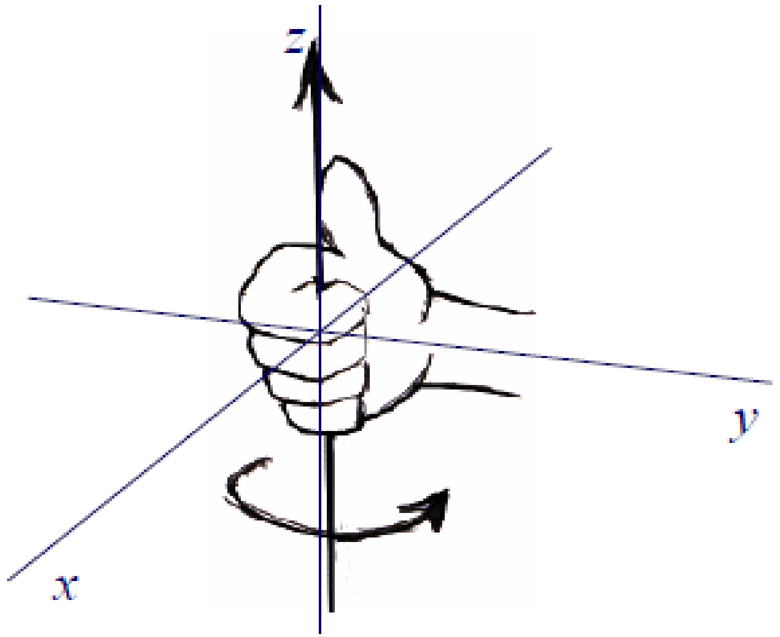
1. Which of these show the proper projection of the vector onto the z axis?



A. I.
B. II.
C. III.
D. IV.
E. None of the above.

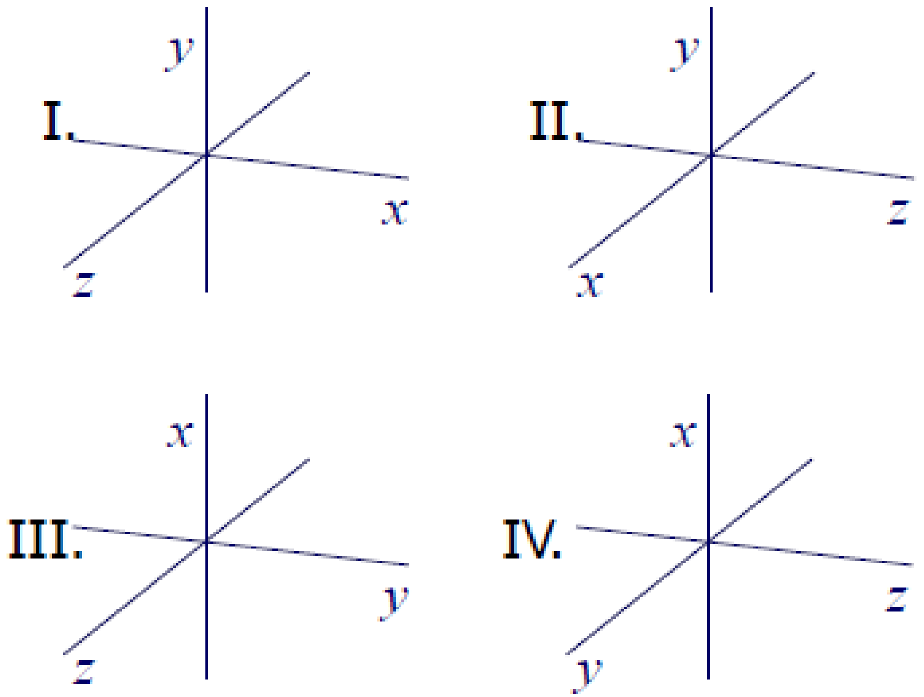
A Note About Right-Hand Coordinate Systems

- A three-dimensional coordinate system MUST obey the right-hand rule.
- Curl the fingers of your RIGHT HAND so they go from x to y. Your thumb will point in the z direction.



Right Handed Coordinate Systems

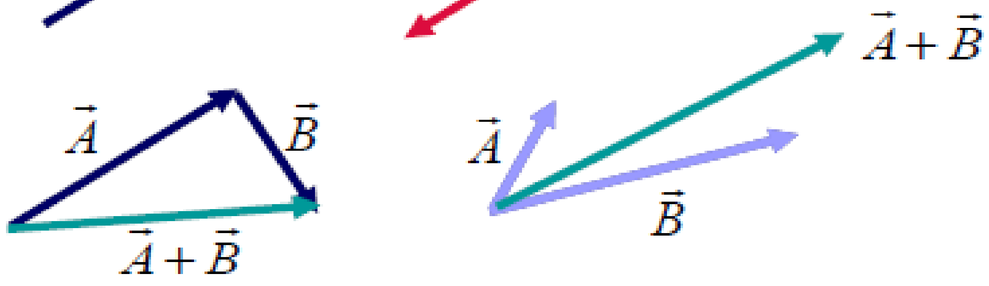
2. Which of these coordinate systems obey the right-hand rule?

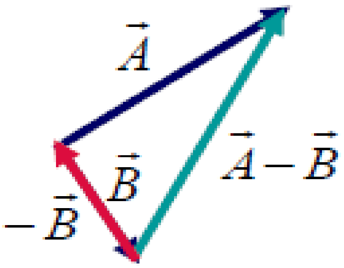


A. I and II.
B. II and III.
C. I, II, and III.
D. I and IV.
E. IV only.

Vector Math

- Vector Inverse
  - Just switch direction
- Vector Addition
  - Use head-tail method, or parallelogram method
- Vector Subtraction
  - Use inverse, then add
- Vector Multiplication
  - Two kinds!
  - Scalar, or dot product
  - Vector, or cross product





Vector Addition by Components

$$\vec{A}+\vec{B}=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

Projection of a Vector: Dot Product

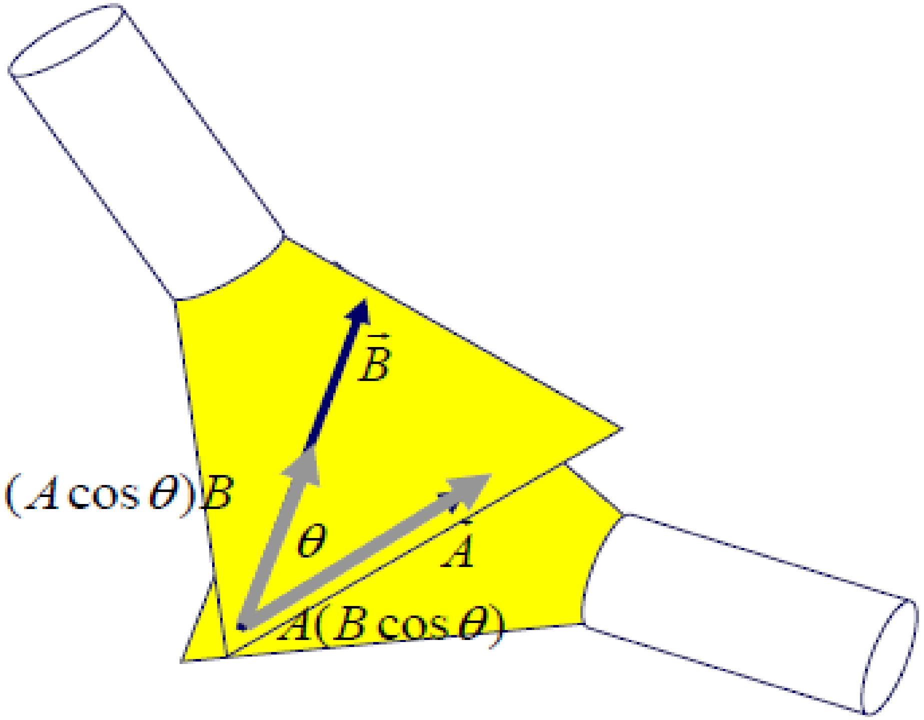
- The dot product says something about how parallel two vectors are.
- The dot product (scalar product) of two vectors can be thought of as the projection of one onto the direction of the other.

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\hat{i} = A\cos\theta = A_x$$

- Components

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$



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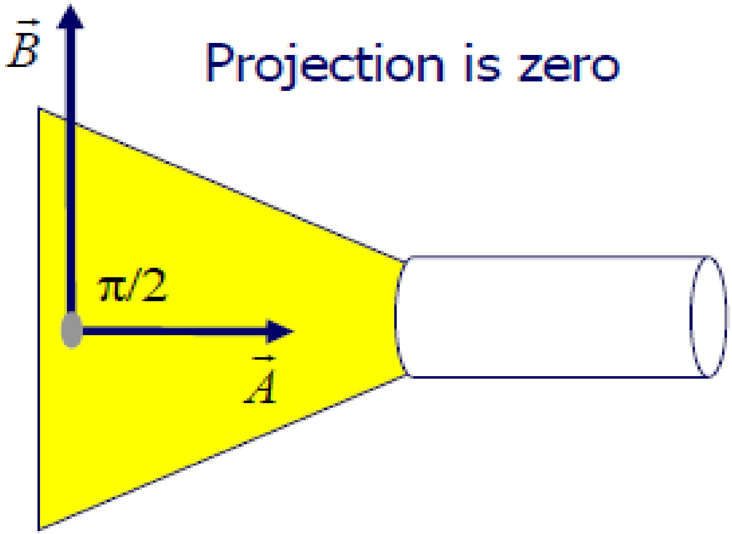


Derivation

- How do we show that $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$?
- Start with
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then
$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
- But
$$\hat{i}\cdot\hat{j} = 0;\ \hat{i}\cdot\hat{k} = 0;\ \hat{j}\cdot\hat{k} = 0$$
$$\hat{i}\cdot\hat{i} = 1;\ \hat{j}\cdot\hat{j} = 1;\ \hat{k}\cdot\hat{k} = 1$$
- So
$$\vec{A}\cdot\vec{B} = A_x\hat{i}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_y\hat{j} + A_z\hat{k}\cdot B_z\hat{k}$$
$$= A_xB_x + A_yB_y + A_zB_z$$

Cross Product

- The cross product of two vectors says something about how perpendicular they are. You will find it in the context of rotation, or twist.

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$

- Direction perpendicular to both A and B (right-hand rule) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
- Components (messy)

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

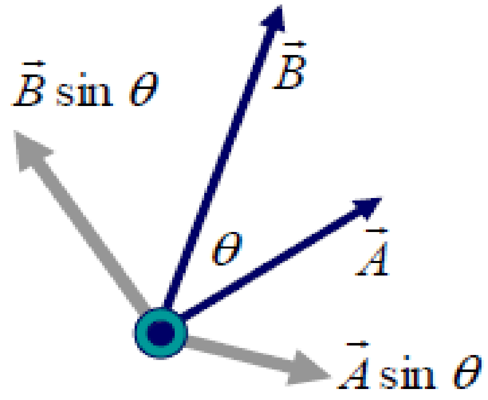
$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

- Recall angular momentum

$$\vec{L} = \vec{r}\times\vec{p}$$

- Torque

$$\vec{\tau} = \vec{r}\times\vec{F}$$



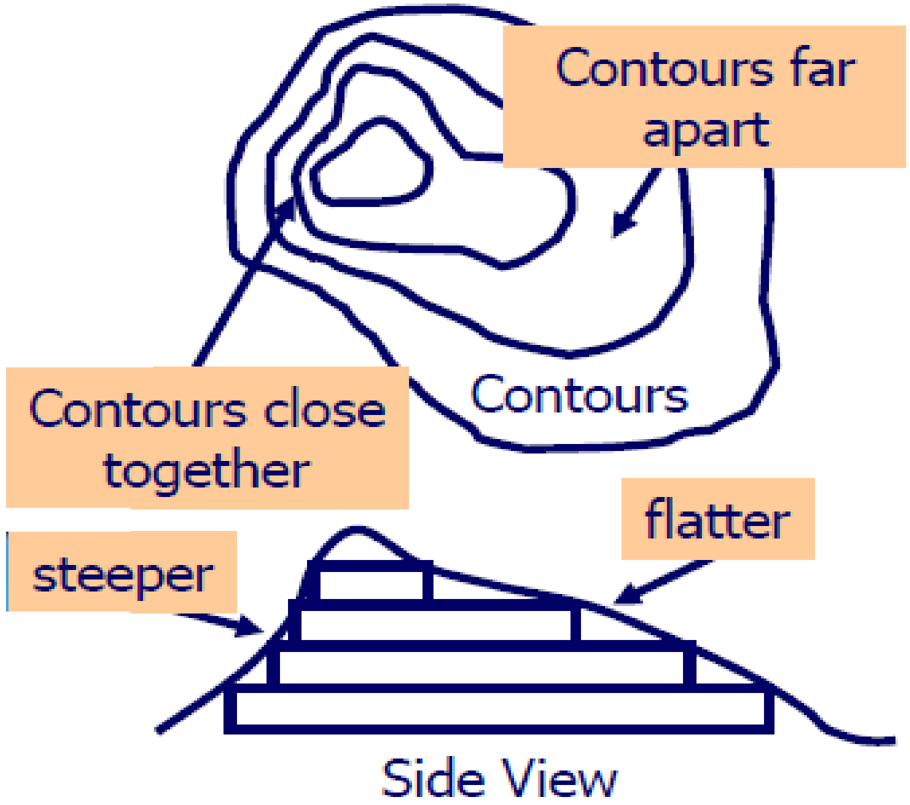
Derivation

- How do we show th at $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$?
- Start with $$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$ $$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then $$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_x\hat{i}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\end{aligned}$$
- But $$\hat{i}\times\hat{j} = \hat{k};\ \hat{i}\times\hat{k} = -\hat{j};\ \hat{j}\times\hat{k} = \hat{i}$$ $$\hat{i}\times\hat{i} = 0;\ \hat{j}\times\hat{j} = 0;\ \hat{k}\times\hat{k} = 0$$
- So $$\begin{aligned}\vec{A}\times\vec{B} &= A_x\hat{i}\times B_y\hat{j} + A_x\hat{i}\times B_z\hat{k} + A_y\hat{j}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_z\hat{k} \\ &\quad + A_z\hat{k}\cdot B_x\hat{i} + A_z\hat{k}\cdot B_y\hat{j}\end{aligned}$$

Scalar Fields

- A scalar field is just one where a quantity in "space" is represented by numbers, such as this temperature map.

- Here is another scalar field, height of a mountain.



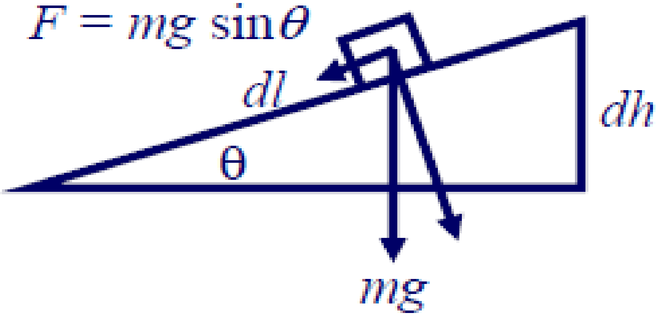
Gradients and Gravity

- Height contours h, are proportional to potential energy $U = mgh$. If you move along a contour, your height does not change, so your potential energy does not change.
- If you move downhill, on say a 6% grade, it means the slope is 6/100 (for every 100 m of horizontal motion, you move downward by 6 m).

- Grade and gradient mean the same thing. A 6% grade is a gradient of

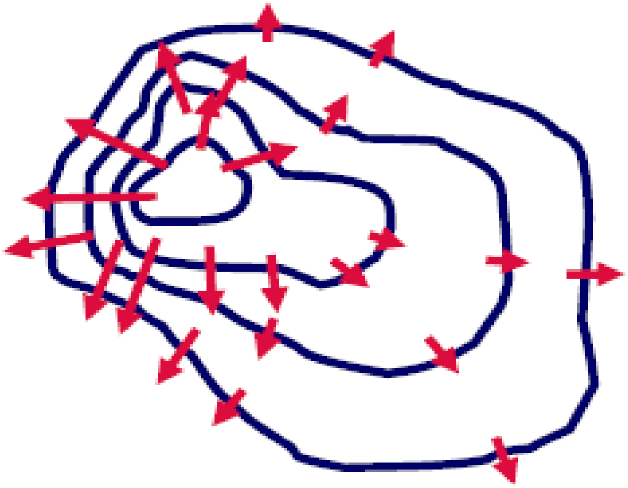
$$\lim_{\Delta x \to 0} \Delta h / \Delta x = dh / dx = -0.06$$

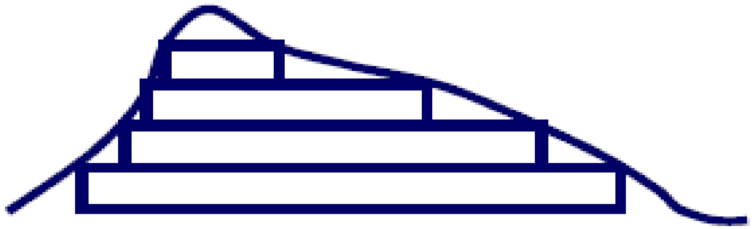
$$F = -dU / dl = -d\, mgh / dl = -mg\, dh / dl$$



Vector Fields

- A vector field is one where a quantity in "space" is represented by both magnitude and direction, i.e by vectors.
- The vector field bears a close relationship to the contours (lines of constant potential energy).
- The steeper the gradient, the larger the vectors.
- The vectors point along the direction of steepest descent, which is also perpendicular to the lines of constant potential energy.
- Imagine rain on the mountain. The vectors are also "streamlines." Water running down the mountain will follow these streamlines.

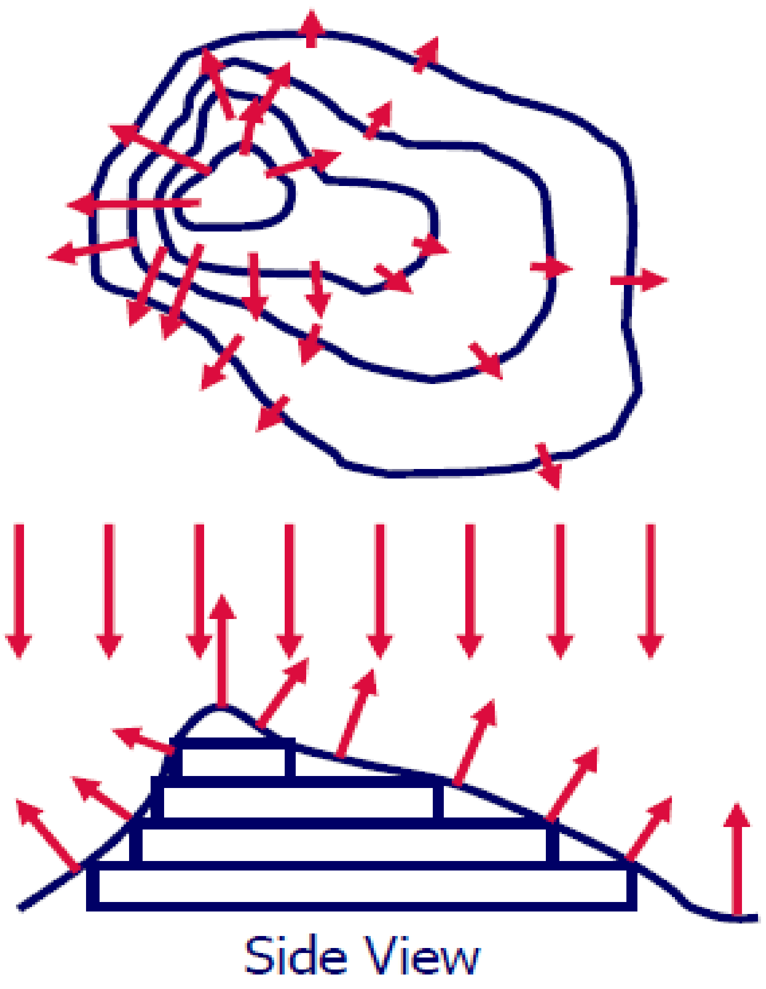




Side View

Surface vs. Volume Vector Fields

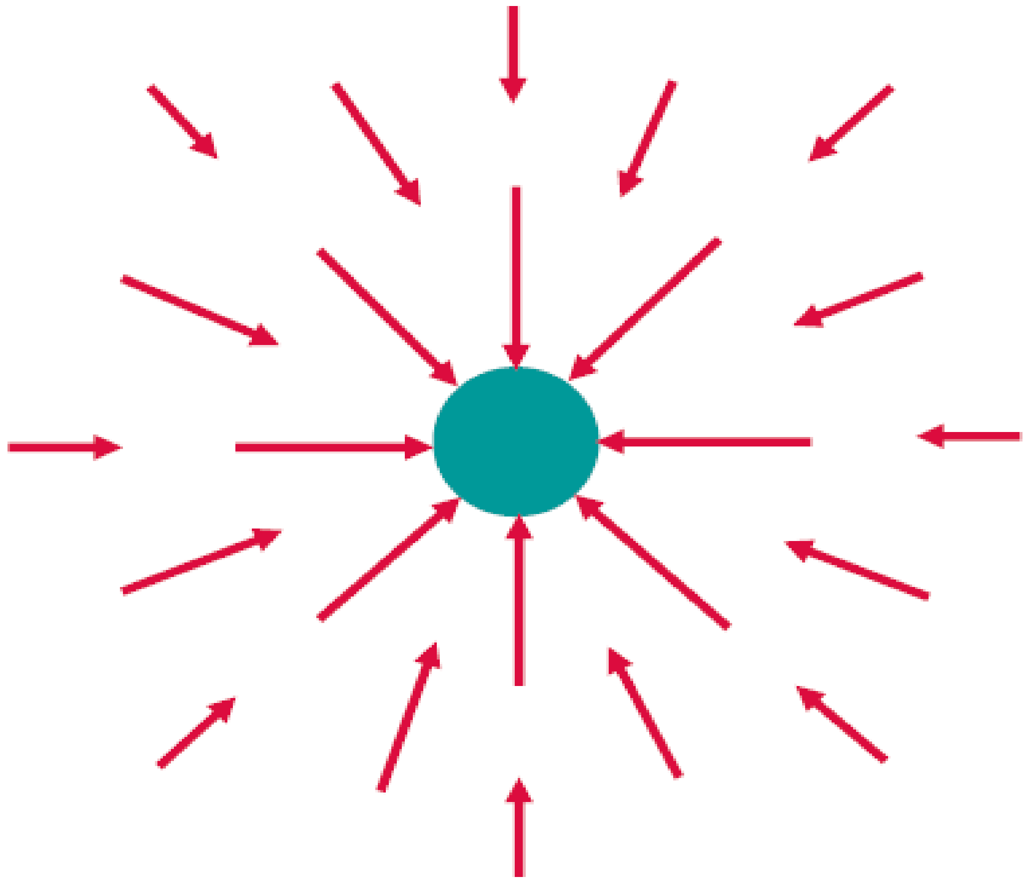
- In the example of the mountain, note that these force vectors are only correct when the object is ON the surface.
- The actual force field anywhere other than the surface is everywhere downward (toward the center of the Earth.
- The surface creates a "normal force" everywhere normal (perpendicular) to the surface.
- The vector sum of these two forces is what we are showing on the contour plot.



Vector Field Due to Gravity

- When you consider the force of Earth's gravity in space, it points everywhere in the direction of the center of the Earth. But remember that the strength is:

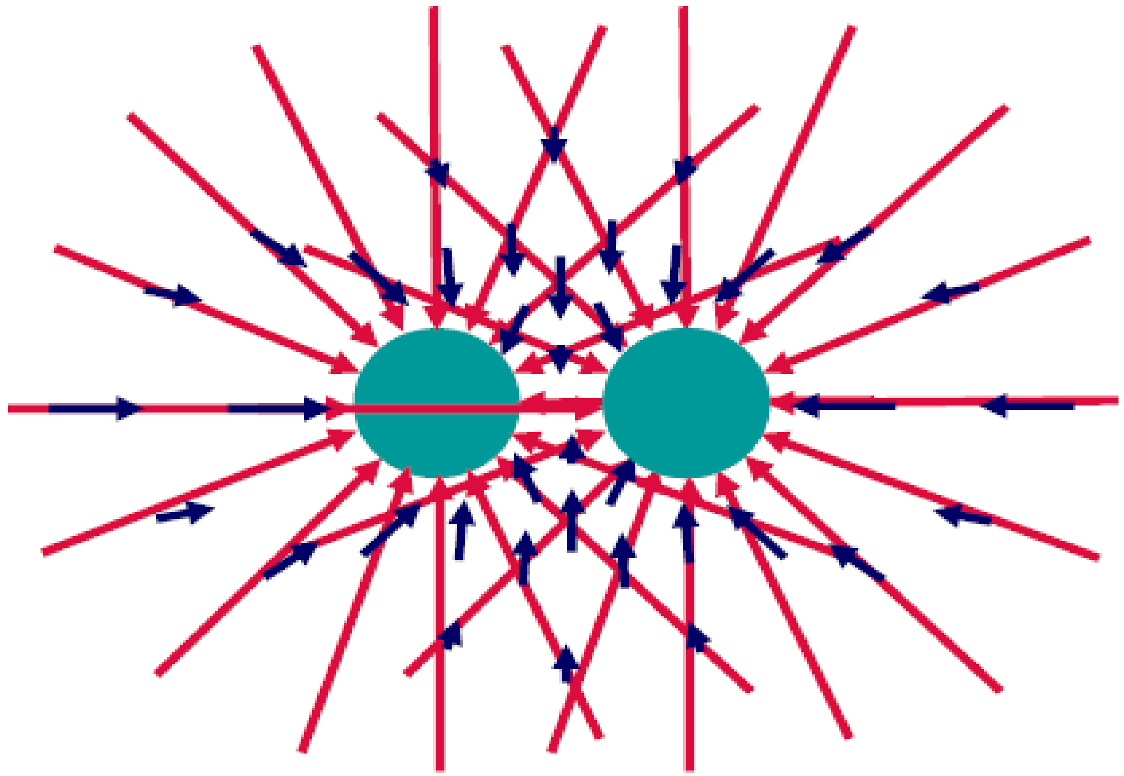
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$



- This is an example of an inverse-square force (proportional to the inverse square of the distance).

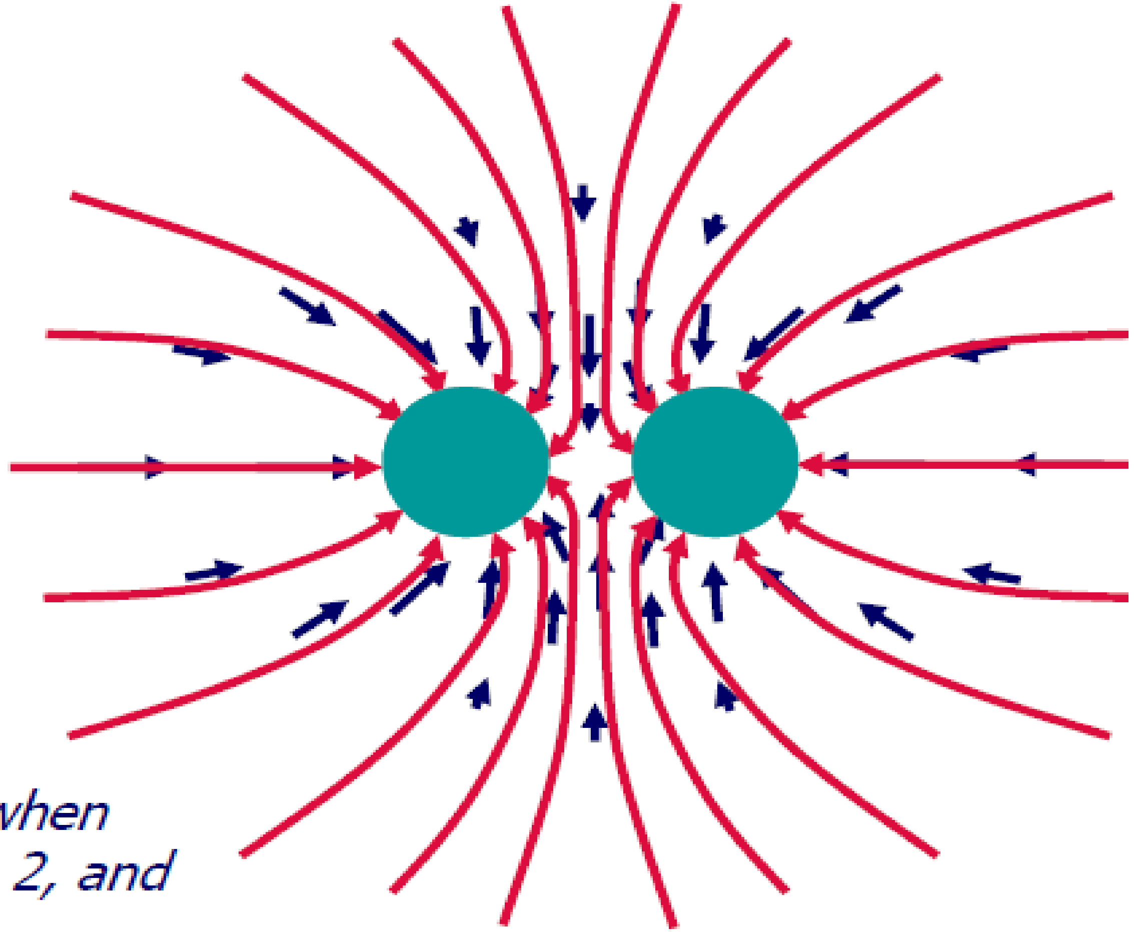
Gravitational Field

- We can therefore think of the "action-at-a-distance" of gravity as a field that permeates all of space.
- We draw "field lines" that show both the direction and strength of the field (from the density of field lines).
- The field cannot be seen or touched, and has no effect until you consider a second mass.
- What happens if we have two equal masses? Superposition—just vector sum the two fields.



Gravitational Field of Two Equal Masses

- Again, think of adding a small test mass.
- The force vectors show the direction and strength of the force on such a test mass.
- We can draw field lines that follow the force vectors.



We will be using these same concepts when we talk about electric charge in Lecture 2, and the electric field in Lecture 3.