ENE 503 – Computational Fluid Dynamics

WEEK 3: DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS:

- **Ordinary differential equation:**

- Ordinary differential equation (ODE): an equation which, other than the one independent variable *x* and the dependent variable *y*, also contains derivatives from *y* to *x*. General form

$$F(x,y,y',y'' \ldots y^{(n)}) = 0$$

here *n* is the highest order derivative and the order of the equation is determined by the order *n* of the

- A partial differential equation (PDE) has two or more independent variables. A PDE with two independent variables has the following form:

$$F\left(x,\ y,\ z,\ \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial x \partial y},\ \frac{\partial^2 z}{\partial y^2},\ \ldots\right) = 0$$

with *z=z(x,y).*

- the order of the highest order partial derivative in the equation determines the order here

- **A general partial differential equation in coordinates *x* and *y*:**

$$a\frac{\partial^2 \varphi}{\partial x^2} + b\frac{\partial^2 \varphi}{\partial x \partial y} + c\frac{\partial^2 \varphi}{\partial y^2} + d\frac{\partial \varphi}{\partial x} + e\frac{\partial \varphi}{\partial y} + f\varphi + g = 0$$

➢ where the coefficients *a*, *b*, *c*, *d*, *e*, *f* and *g* are in general functions of the dependent variable, φ and the independent variables *x*, and *y*.

- Any solution to the above equation represents a surface in space

- The first derivatives of the above equation are continuous functions of the *x* and *y.* The total differentials*:*

$$d\varphi_x = \frac{\partial \varphi_x}{\partial x}dx + \frac{\partial \varphi_x}{\partial y}dy = \frac{\partial^2 \varphi}{\partial x^2}dx + \frac{\partial^2 \varphi}{\partial x \partial y}dy$$

$$d\varphi_y = \frac{\partial \varphi_y}{\partial x}dx + \frac{\partial \varphi_y}{\partial y}dy = \frac{\partial^2 \varphi}{\partial x \partial y}dx + \frac{\partial^2 \varphi}{\partial y^2}dy$$

- The original differential equation can be expressed as

$$a\frac{\partial^2 \varphi}{\partial x^2} + b\frac{\partial^2 \varphi}{\partial x \partial y} + c\frac{\partial^2 \varphi}{\partial y^2} = -d\frac{\partial \varphi}{\partial x} - e\frac{\partial \varphi}{\partial y} - f\varphi - g = h$$

The last three equations above form of a system of three linear equations with three unknowns, $\frac{\partial^2 \varphi}{\partial x^2}$, $\frac{\partial^2 \varphi}{\partial x \partial y}$, and $\frac{\partial^2 \varphi}{\partial y^2}$. The matrix solution can be provided as below:

$$\begin{bmatrix} a & b & c \\ dx & dy & dz \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \varphi}{\partial x^2} \\ \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial y^2} \end{bmatrix} = \begin{bmatrix} h \\ d\varphi_x \\ d\varphi_y \end{bmatrix}$$

by using Cramer's rule

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\begin{vmatrix} h & b & c \\ d\varphi_x & dy & 0 \\ d\varphi_y & dx & dy \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

$$\frac{\partial^2\varphi}{\partial x\partial y}=\frac{\begin{vmatrix} a & b & h \\ dx & d\varphi_x & 0 \\ 0 & d\varphi_y & dy \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

$$\frac{\partial^2\varphi}{\partial y^2}=\frac{\begin{vmatrix} a & b & h \\ dx & dy & 0 \\ 0 & dx & d\varphi_y \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

The second order derivatives ıf the dependent variables along the characteristics when these derivatives are indeterminant. The denominator of last three equations above must be zero.

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

This equation can be written as

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

The slope of the above equation can be determined as:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Characterization depends on the roots of the higher order terms (second order terms):

  - Hyperbolic nature when $b^2 - 4ac > 0$
  - Parabolic nature when $b^2 - 4ac > 0$
  - Elliptic behavior when b2 – 4ac < 0

- **Origin of the terms:**

- The "elliptic," "parabolic," or "hyperbolic terms are used to label these equations is simply a direct analogy with the case for conic sections.

- The general equation for a conic section from analytic geometry is:

  where if

  - $b^2 - 4ac > 0$ the conic is a hyperbola.
  - $b^2 - 4ac = 0$ the conic is a parabola.
  - $b^2 - 4ac > 0$ the conic is an ellipse.

References:

1. Aksel, M.H., 2016, "Notes on Fluids Mechanics", Vol. 1, METU Publications
2. Versteeg H.K., and W. Malalasekera V., 1995, "Computational Fluid Dynamics: The Finite Volume Method", Longman Scientific & Technical, ISBN 0-582-21884-5