

ENE 505 – Applied Computational Fluid Dynamics in Renewable Energy Technologies

WEEK 3: DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS:

- **Ordinary differential equation:**

- Ordinary differential equation (ODE): an equation which, other than the one independent variable x and the dependent variable y , also contains derivatives from y to x . General form

$$F(x, y, y', y'' \dots y^{(n)}) = 0$$

here n is the highest order derivative and the order of the equation is determined by the order n of the

- A partial differential equation (PDE) has two or more independent variables. A PDE with two independent variables has the following form:

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \dots\right) = 0$$

with $z=z(x,y)$.

- the order of the highest order partial derivative in the equation determines the order here

- **A general partial differential equation in coordinates x and y :**

$$a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d \frac{\partial \phi}{\partial x} + e \frac{\partial \phi}{\partial y} + f\phi + g = 0$$

- where the coefficients a, b, c, d, e, f and g are in general functions of the dependent variable, φ and the independent variables x , and y .
- Any solution to the above equation represents a surface in space
- The first derivatives of the above equation are continuous functions of the x and y .
The total differentials:

$$d\varphi_x = \frac{\partial \varphi_x}{\partial x} dx + \frac{\partial \varphi_x}{\partial y} dy = \frac{\partial^2 \varphi}{\partial x^2} dx + \frac{\partial^2 \varphi}{\partial x \partial y} dy$$

$$d\varphi_y = \frac{\partial \varphi_y}{\partial x} dx + \frac{\partial \varphi_y}{\partial y} dy = \frac{\partial^2 \varphi}{\partial x \partial y} dx + \frac{\partial^2 \varphi}{\partial y^2} dy$$

- The original differential equation can be expressed as

$$a \frac{\partial^2 \varphi}{\partial x^2} + b \frac{\partial^2 \varphi}{\partial x \partial y} + c \frac{\partial^2 \varphi}{\partial y^2} = -d \frac{\partial \varphi}{\partial x} - e \frac{\partial \varphi}{\partial y} - f\varphi - g = h$$

The last three equations above form of a system of three linear equations with three unknowns, $\frac{\partial^2 \varphi}{\partial x^2}$, $\frac{\partial^2 \varphi}{\partial x \partial y}$, and $\frac{\partial^2 \varphi}{\partial y^2}$. The matrix solution can be provided as

below:

$$\begin{bmatrix} a & b & c \\ dx & dy & dz \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \varphi}{\partial x^2} \\ \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial y^2} \end{bmatrix} = \begin{bmatrix} h \\ d\varphi_x \\ d\varphi_y \end{bmatrix}$$

by using Cramer's rule

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\begin{vmatrix} h & b & c \\ d\varphi_x & dy & 0 \\ d\varphi_y & dx & dy \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\begin{vmatrix} a & b & h \\ dx & d\varphi_x & 0 \\ 0 & d\varphi_y & dy \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{\begin{vmatrix} a & b & h \\ dx & dy & 0 \\ 0 & dx & d\varphi_y \end{vmatrix}}{\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

The second order derivatives of the dependent variables along the characteristics when these derivatives are indeterminate. The denominator of last three equations above must be zero.

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

This equation can be written as

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

The slope of the above equation can be determined as:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

➤ Characterization depends on the roots of the higher order terms (second order terms):

- Hyperbolic nature when $b^2 - 4ac > 0$
- Parabolic nature when $b^2 - 4ac = 0$
- Elliptic behavior when $b^2 - 4ac < 0$

• **Origin of the terms:**

➤ The “elliptic,” “parabolic,” or “hyperbolic terms are used to label these equations is simply a direct analogy with the case for conic sections.

➤ The general equation for a conic section from analytic geometry is:

where if

- $b^2 - 4ac > 0$ the conic is a hyperbola.
- $b^2 - 4ac = 0$ the conic is a parabola.
- $b^2 - 4ac < 0$ the conic is an ellipse.

References:

1. Aksel, M.H., 2016, “Notes on Fluids Mechanics”, Vol. 1, METU Publications
2. Versteeg H.K., and W. Malalasekera V., 1995, “Computational Fluid Dynamics: The Finite Volume Method”, Longman Scientific & Technical, ISBN 0-582-21884-5