LAPLACE TRANSFORMS
SOLUTION OF DIFFERENTIAL EQUATION[1-5]

References:

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The definition of Laplace transforms

$$f(s) = \int_0^{\infty} f(t) e^{-st} dt$$

f(s) is the Laplace transform of the function f(t)

Variable t is eliminated through the variable s at the end of the Laplace transform.

$$s = a + bj$$

Laplace transforms are always applied to linear functions

$$L\left\{A\,f_1(t) + B\,f_2(t)\right\} = A*L\,\{f_1\,(t)\,\} + B*L\,\{f_2(t)\}$$

Laplace transform of equations are factored according to three root types. These are real, complex, and repeated roots.

$$L\{t \sin 3t\} = \frac{(2*3)s}{(s^2+3^2)^2} = \frac{6s}{(s^2+3^2)^2}$$

With the help of translation of transform theorem

$$L\{e^{-at} f(t)\} = f(s+a)$$

$$L\{e^{-2t} t \sin(3t)\} = \frac{6(s+2)}{((s+2)^2+3^2)^2}$$

$$L\{5e^{-2t} t \sin(3t)\} = \frac{30(s+2)}{((s+2)^2+3^2)^2}$$

Solve the differential equation.

$$2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = u(t)$$

$$x(0) = x'(0) = 0$$

$$2\left[s^2x(s) - sx(0) - x'(0)\right] + 2[sx(s) - x(0)] + x(s) = \frac{1}{s}$$

$$2s^2x(s) + 2sx(s) + x(s) = \frac{1}{s}$$

$$x(s) = \frac{1}{s(2s^2 + 2s + 1)}$$

Factoring and expanding in partial fractions, Eq.1 is

$$x(s) = \frac{1}{s(2s^2 + 2s + 1)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + 2s + 1}$$

To determine A, multiply both sides of Eq.(1) by s and set s to 0.

$$A = \left[\frac{s}{s(2s^2+2s+1)}\right]_{s=0}$$

$$A = 1$$

$$x(s) = \frac{1}{s(2s^2+2s+1)} = \frac{1}{s} + \frac{Bs+C}{2s^2+2s+1}$$

the denominator on the left-hand side, $1 = 2s^2 + 2s + 1 + Bs^2 + Cs$

$1 = (2+B)s^2 + (2+C)s + 1$

s^2 : $2 + B = 0 \rightarrow$ $B = -2$

s : $2 + C = 0 \rightarrow$ $C = -2$

$$x(s) = \frac{1}{s} + \frac{-2s-2}{2s^2+2s+1}$$

$$x(s) = \frac{1}{s} - \frac{s+1}{s^2+s+\frac{1}{2}}$$

the quadratic must have the form

$$s^2 + \alpha s + \left(\frac{\alpha}{2}\right)^2 = \left(s + \frac{\alpha}{2}\right)^2$$

$$\frac{s+1}{s^2+s+\frac{1}{2}} = \frac{s+1}{\left(s^2+s+\frac{1}{4}\right)+\frac{1}{2}-\frac{1}{4}} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$x(s) = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$L\{e^{-at}\sin(kt)\} = \frac{k}{(s+a)^2+k^2}$$

$$L\{e^{-at}\cos(kt)\} = \frac{s+a}{(s+a)^2+k^2}$$

$$x(s) = \frac{1}{s} - \frac{\left(s + \frac{1}{2}\right) + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{1}{4}} = \frac{1}{s} - \frac{\left(s + \frac{1}{2}\right)}{\left(s + \frac{1}{2}\right)^2 + \frac{1}{4}} - \frac{\frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{1}{4}}$$

Inverting these terms to obtain the solution to the differential equation, we get

$$x(t) = 1 - e^{-\frac{t}{2}} cos\left(\frac{t}{2}\right) - e^{-\frac{t}{2}} sin\left(\frac{t}{2}\right)$$