SHEAR FORCES AND MOMENTS IN BEAMS

When a beam is loaded by forces and couples, internal stresses arise as normal and shear. In order to determine the magnitude of stresses at any section of the beam, it is necessary to know the resultant force and moment acting at that section by applying the equations of static equilibrium.

Resisting Moment

The magnitude of "M" which states that the sum of moments of all forces about an axis and perpendicular to the plane of the page is zero. The resisting moment M is due to stresses that are distributed over the vertical section.

Resisting Shear

The vertical force is called the resisting shear. For equilibrium of forces in the vertical direction, this force is actually the resultant of shearing stresses distributed over the vertical section on a beam.

Bending Moment

The algebraic sum of the moments of external forces to one side of the vertical section of a beam about an axis. Bending moment is opposite in direction to the resisting moment with same magnitude. Bending moment rather than the resisting moment is used in calculations because it can be represented directly in terms of the external loads.

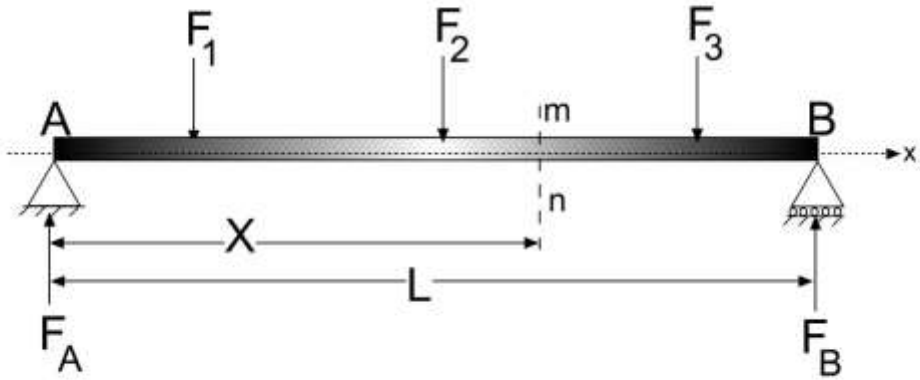
Shear Force

The algebraic sum of all the vertical forces to one side is called the shearing force at that section. Shear force is opposite in direction to the resisting shear but of the same magnitude. It is ordinarily used in calculations, rather than the resisting shear.

Concentrated (Point) Loads on Beams

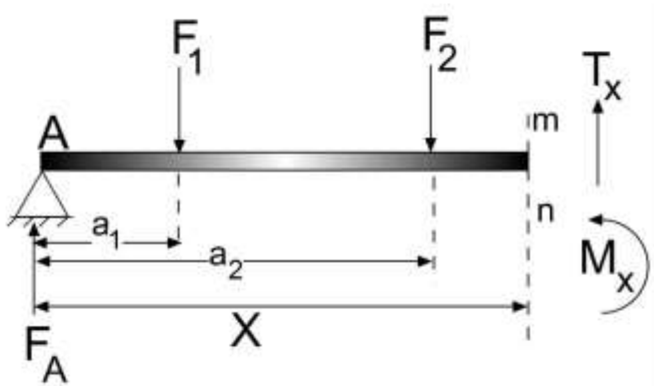
In case a single beam is loaded, "***Shear Forces***" will act on supports in opposite direction. Any shear force from "A" point in "X" distance turns out to be;

$$T_x = F_A - F_1 - F_2$$

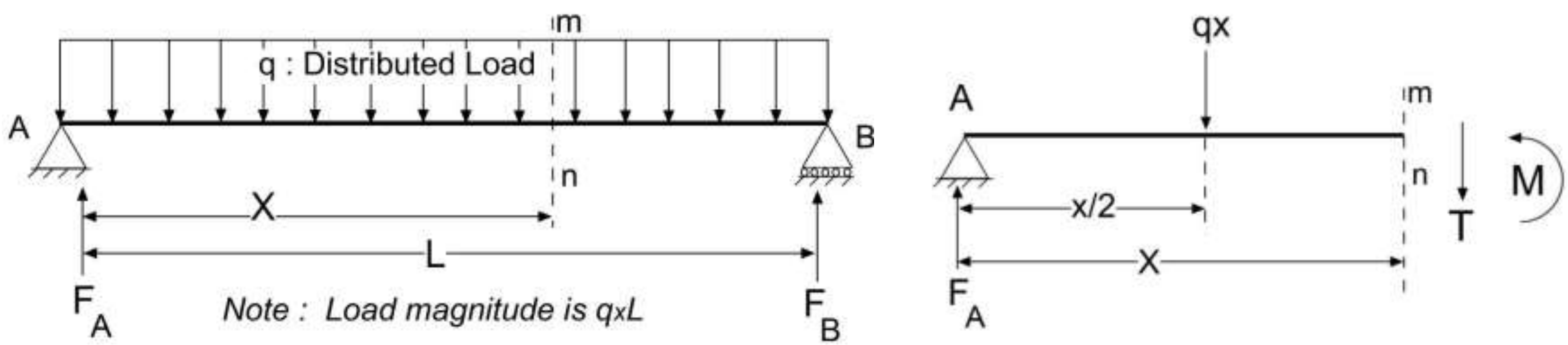


Moreover, due to the magnitude(s) of load(s), the beam might be subjected to "***Bending Moment***" on "m-n" cross section that is;

$$M_x = F_A X - F_1(X - a_1) - F_2(X - a_2)$$



DISTRIBUTED LOADS ON BEAMS



Support Reactions at “A” and “B” are equal;

$$F_B=(qL/2) = F_A$$

Shear Force on “mn” cross section is;

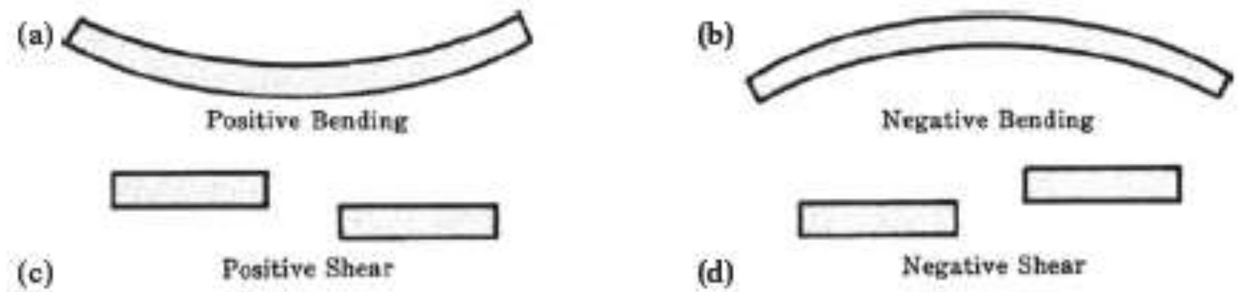
$$T_x=F_A-qX=(qL/2)-qX = q(1/2-X)$$

Bending Moment is;

$$M_x=F_AX-(qX^2/2) = (qL/2)X-(qX^2/2) = (qX/2)(1-X)$$

SIGN CONVENTIONS

Due to the magnitude, type and location of loading, beams tend to bend in upward or downward direction.
Figure (a): Concave bending (say in direction of gravity) of the beam . Positive bending leads to produce Positive Bending.
Figure (c): Left portion of the beam is sheared upwards with respect to right portion is “Positive Shear”



BEAMS IN BENDING

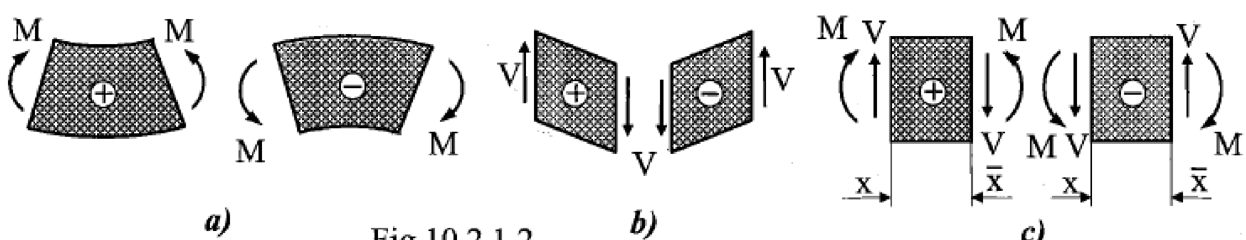
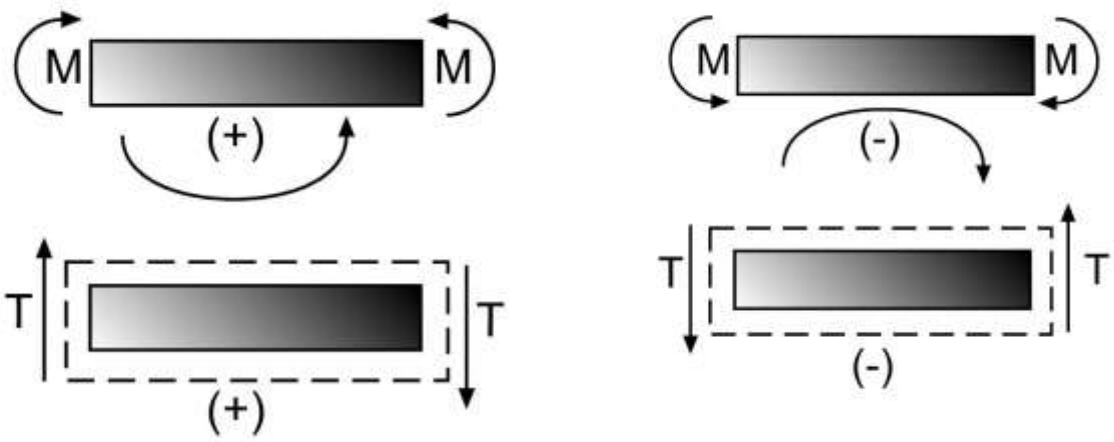


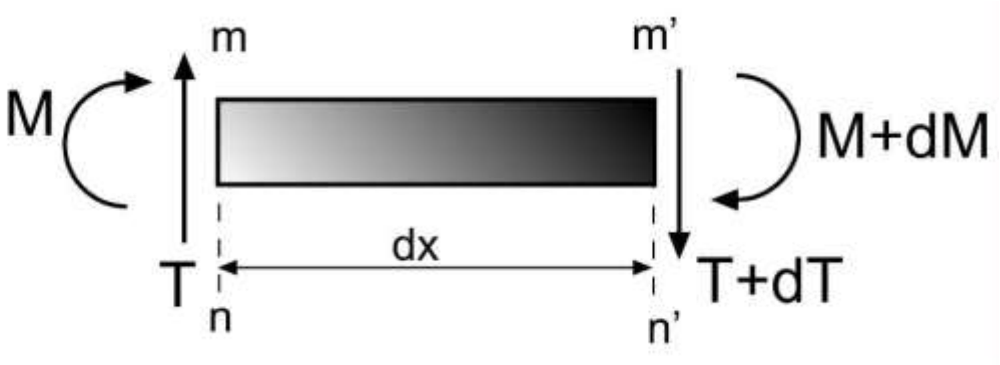
Fig.10.2.1.2



If the bending moment are positive, forces at the left of mn cross-section have CW and forces on the right of mn have CCW moment directions.

RELATION BETWEEN SHEAR FORCE and BENDING MOMENTS

SINGLE LOAD : Shear forces are equal, however moments are different **if there exist no additional forces** between mn and m'n' sections.

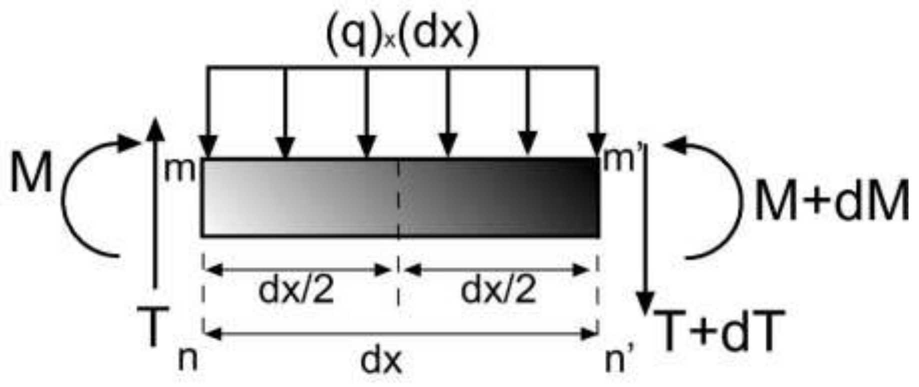


$$\Sigma M_{m'n'} = 0 \; ; \; M+T\,dx-(M+dM)=0 \; ;$$

$$\mathbf{T=dM/dx}$$

Shear force for every single parts of beams between forces are the derivative of bending moment

DISTRIBUTED LOAD : Derivative of shear force equals to the negative magnitude of distributed load without **any additional loading** between mn and m'n' cross sections.



Resultant of vertical loads must be zero; $\Sigma x = 0$; $T\text{-}q\ dx - (T+dT)=0$;

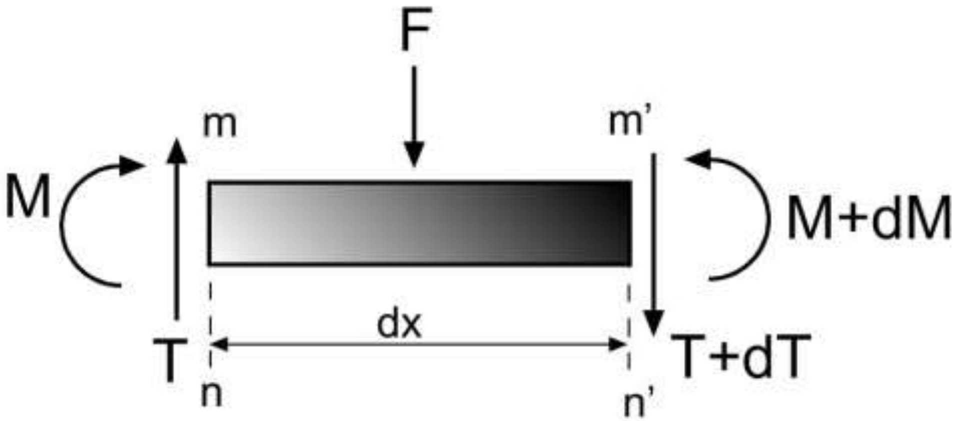
$$q=(-)dT/dx$$

Moment on m'n' section must be zero; $M-(M+dM)+T\ dx - (q\ dx^2/2)=0$;

$$T=dMT/dx$$

Note: dx is negligible, $dx^2/2$ is even smaller to be accepted as zero

<u>SINGLE LOAD:</u> In case there is another “F” load between mn and m’n’, the shear force will be different by the magnitude of “F” just nearby the application point of the “F” force. Due, the derivative of (*dM/dx)* will also change.



$$-q = dT/dx = d^2M/dx^2$$

$$-\int q \, dx = (dM/dx) = T$$

$$-\int\int q \, dx \, dx = \int T \, dx = M$$

ESSENTIALS of SHEAR FORCE and MOMENT DIAGRAMS

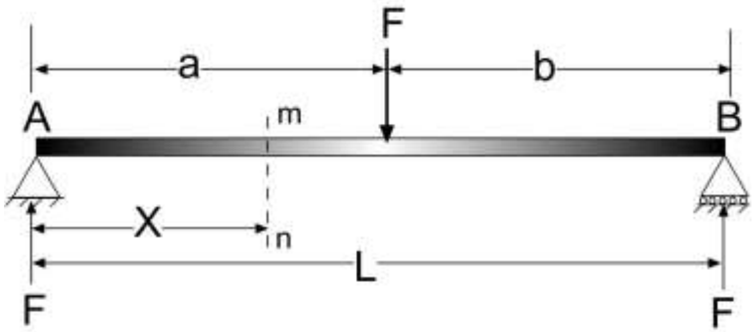
1. Bending moments are minimum or maximum where the slope of bending moments are zero. Minimum and maximum moment values indicate the zero shear forces

2. Variation of shear forces are fixed (constant in value) and variation of moments are linear where the magnitude of distributed loads are zero

3. The variation of shear forces are linear and variation of moments are parabolic for zero slope distributed loading (no angle)

4. In case of triangular loading; shear forces are second and moments are third degree parabolic in shape

ESSENTIALS of SHEAR FORCE and MOMENT DIAGRAMS



Total moment at "B" : $(F_A \times L)-(F \times b)=0$; $F_A=(F \times b/L)$

Total moment at "A" : $F_B \times L-F \times a=0$; $F_B=(F \times a/L)$

Section I :

Moment about mn section

$-F_A \times X = M_x$; $M_x = [(F \times b)/L]x$

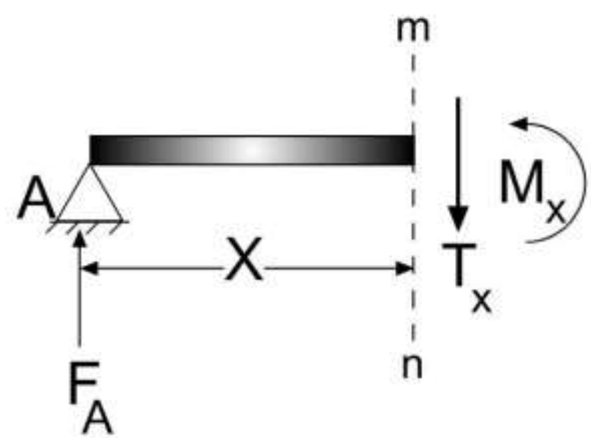
Shear Force;

$T_x = (dM_x/dx) = F_A = (F \times b)/L$

Boundary Conditions

$X = 0$; $M_x= [(F \times b)/L]x =0$ and $T_x=(F \times b)/L$

$X=a$; $M_x = [(F \times b)/L]a$ and $T_x = (F \times b)/L$



Section II :

Moment about mn section

$M'x = F_A \times X' - F(X'-a) = [(F \times b)/L]x' - F(X'-a)$

Boundary Conditions

$X' = a;\ M = [(F \times b)/L]a - F(a-a) = (F \times b \times a)/L$

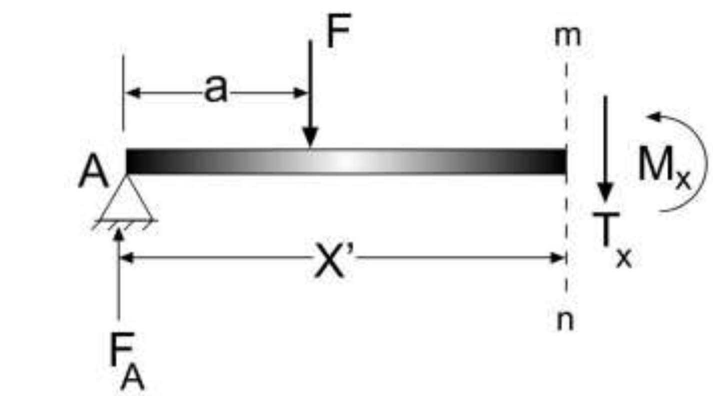
$X' = L;\ M = [(F \times b)/L]L - (F(L-a) = F \times b - F \times b = 0$

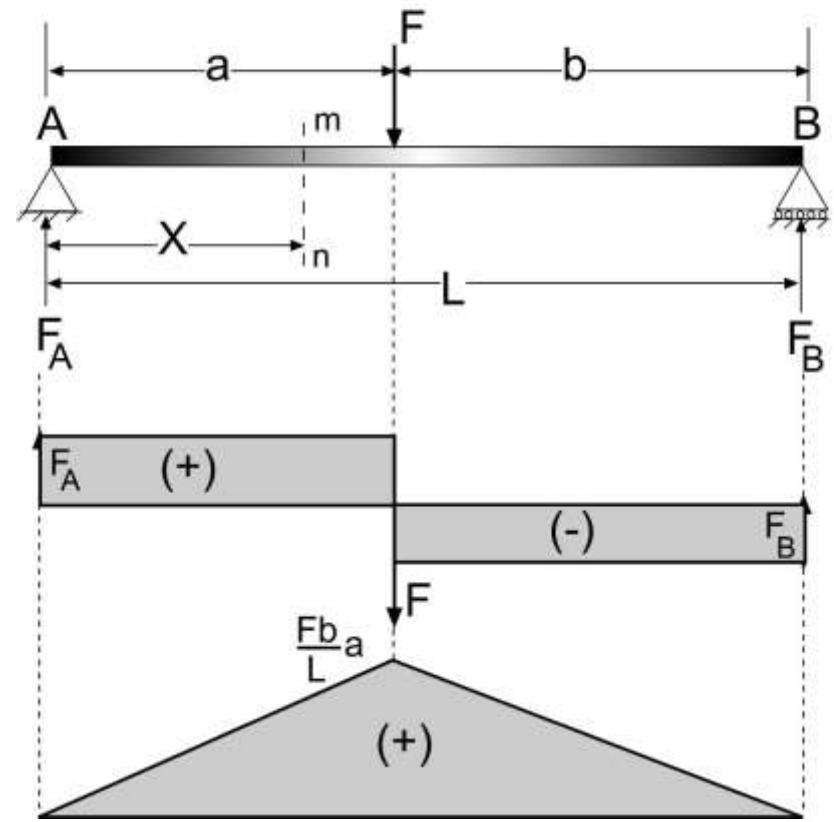
Shear Force;

$T_{x'} = (M_x'/dx') = [(F \times b)/L] - F = [(F \times b - F \times L)/L] = -F(a/L)$

The area of shear force and moment diagram parts are equal, their total should be zero;

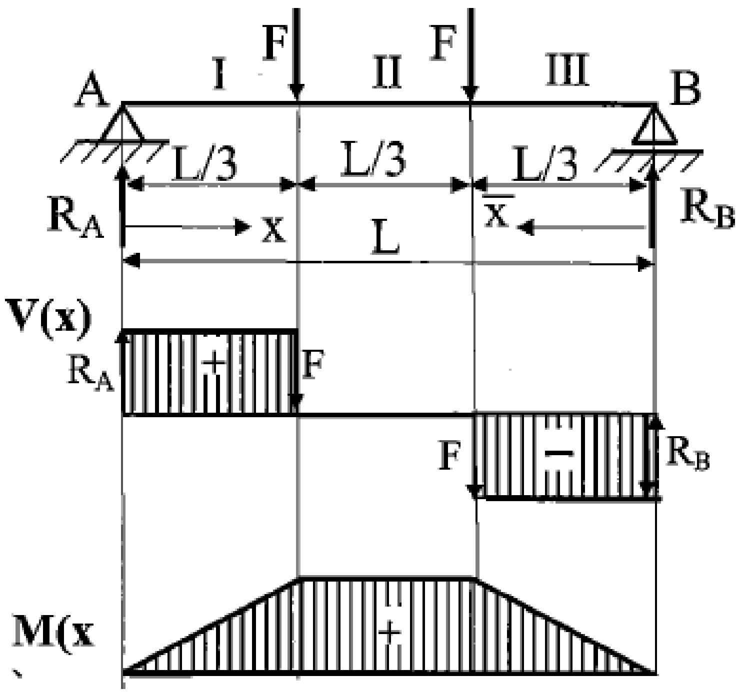
$\int_A^B dM = \int_A^B T dx\ ;\ \int_A^B dM = M_B - M_A$





Shear and Moment Diagram

SYMMETRIC SINGLE LOADING



1) Solution by applying the ***method of sections***:

<u>Reactions.</u>

$$M_B: \quad R_A \cdot L - F \cdot \frac{2}{3} L - F \cdot \frac{1}{3} L = 0 \quad \Rightarrow \quad R_A = F$$

$$M_A: \quad R_B \cdot L - F \cdot \frac{2}{3} L - F \cdot \frac{1}{3} L = 0 \quad \Rightarrow \quad R_B = F$$

Note: This result can be obtained simply from the symmetry.

<u>Shearing forces and bending moments.</u>

Proceeding from the left:

I. $(0 < x < L/3)$: $V_I(x) = R_A = F$; $\quad M_I(x) = R_A \cdot x = F \cdot x$

II. $(L/3 < x < 2L/3)$: $V_{II}(x) = R_A - F = 0$;

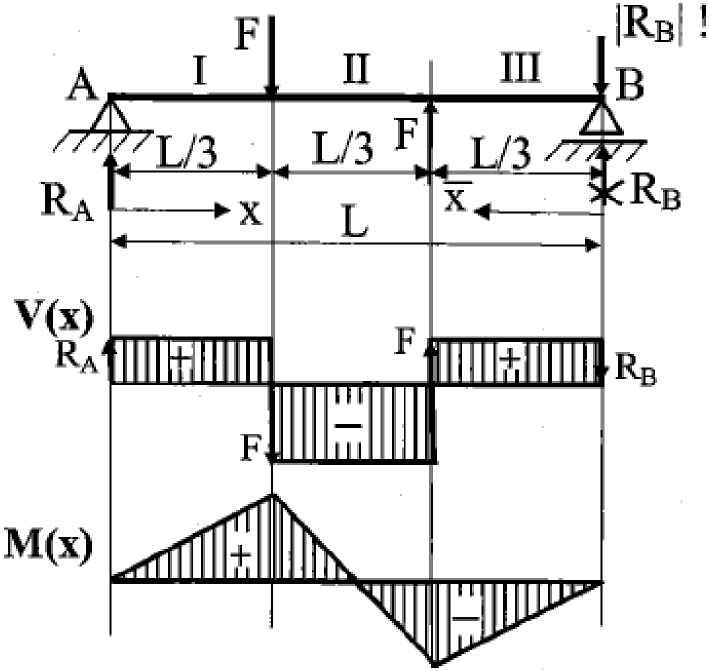
$M_{II}(x) = R_A \cdot x - F \cdot (x - L/3) = F \cdot L/3$

III. $(2L/3 < x < L)$:

$$V_{III}(x) = R_A - F - F = -F;$$

$$M_{III}(x) = R_A \cdot x - F \cdot (x - L/3) - F(x - 2L/3) = F \cdot (L - x)$$

ANTI - SYMMETRIC SINGLE LOADING



The analytical solution by *the **method of sections*** is here again more convenient than that of *the **relations between w, V and M***, while for the graphical illustration, it is conversely.

Reactions.

$$M_B: \quad R_A \cdot L - F \cdot \frac{2}{3} L + F \cdot \frac{1}{3} L = 0 \quad \Rightarrow \quad R_A = \frac{F}{3}$$

$$M_A: \quad R_B \cdot L + F \cdot \frac{2}{3} L - F \cdot \frac{1}{3} L = 0 \quad \Rightarrow \quad R_B = -\frac{F}{3}$$

Shearing forces and bending moments.

Proceeding from the left:

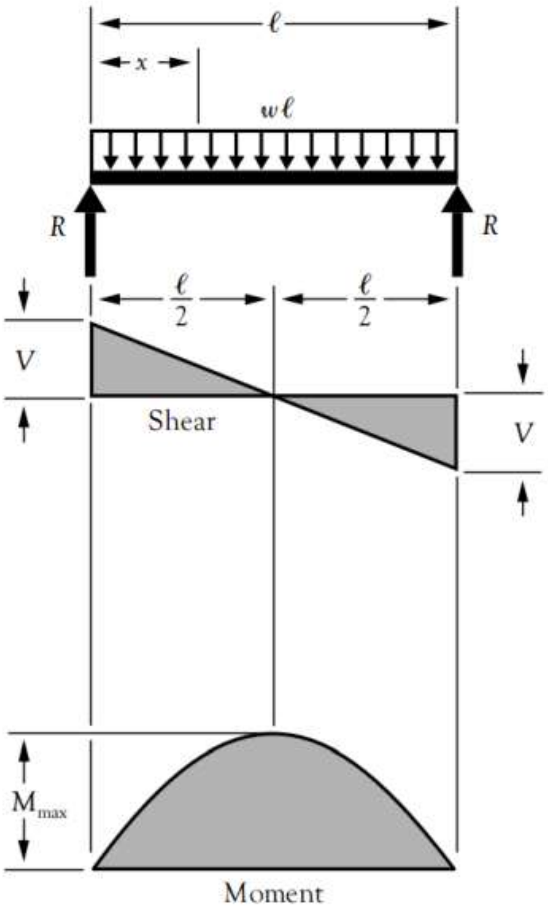
I. $(0 < x < L/3)$: $V_I(x) = R_A = \frac{F}{3}$; $\quad M_I(x) = R_A \cdot x = \frac{F}{3} \cdot x$

II. $(L/3 < x < 2L/3)$: $V_{II}(x) = R_A - F = -\frac{2F}{3}$; $M_{II}(x) = R_A \cdot x - F \cdot \left(x - \frac{L}{3}\right) = \frac{F \cdot L}{3}\left(1 - 2 \cdot \frac{x}{L}\right)$

Proceeding from the right:

III. $(0 < \overline{x} < L/3)$: $V_{III}(\overline{x}) = R_B = \frac{F}{3}$; $\quad M_{III}(\overline{x}) = R_B \cdot \overline{x} = -\frac{F}{3} \cdot \overline{x}$

UNIFORMLY DISTRIBUTED LOADING



$$R = V \quad \dots\dots\dots\dots\dots \quad = \frac{w\ell}{2}$$

$$V_x \quad \dots\dots\dots\dots\dots \quad = w\left(\frac{\ell}{2} - x\right)$$

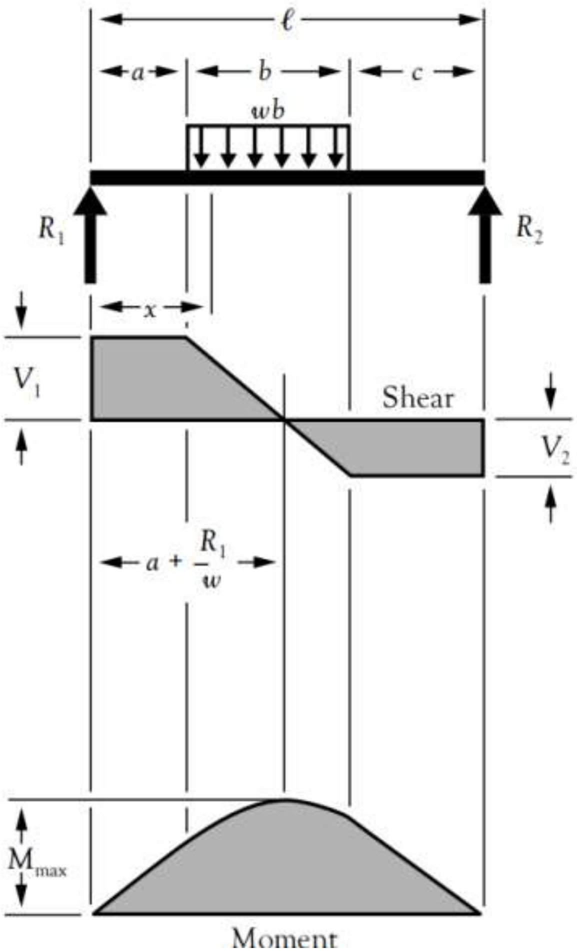
$$M_{max} \text{ (at center)} \quad \dots\dots\dots \quad = \frac{w\ell^2}{8}$$

$$M_x \quad \dots\dots\dots\dots\dots \quad = \frac{wx}{2}(\ell - x)$$

$$\Delta_{max} \text{ (at center)} \quad \dots\dots\dots \quad = \frac{5w\ell^4}{384\ EI}$$

$$\Delta_x \quad \dots\dots\dots\dots\dots \quad = \frac{wx}{24\ EI}(\ell^3 - 2\ell x^2 + x^3)$$

UNIFORMLY DISTRIBUTED PARTIAL LOADING



$$R_1 = V_1 \text{ (max when } a < c) \quad \ldots \quad = \frac{wb}{2\ell}(2c + b)$$

$$R_2 = V_2 \text{ (max when } a > c) \quad \ldots \quad = \frac{wb}{2\ell}(2a + b)$$

$$V_x \left(\text{when } x > a \text{ and } < (a + b)\right) \quad \ldots \quad = R_1 - w(x - a)$$

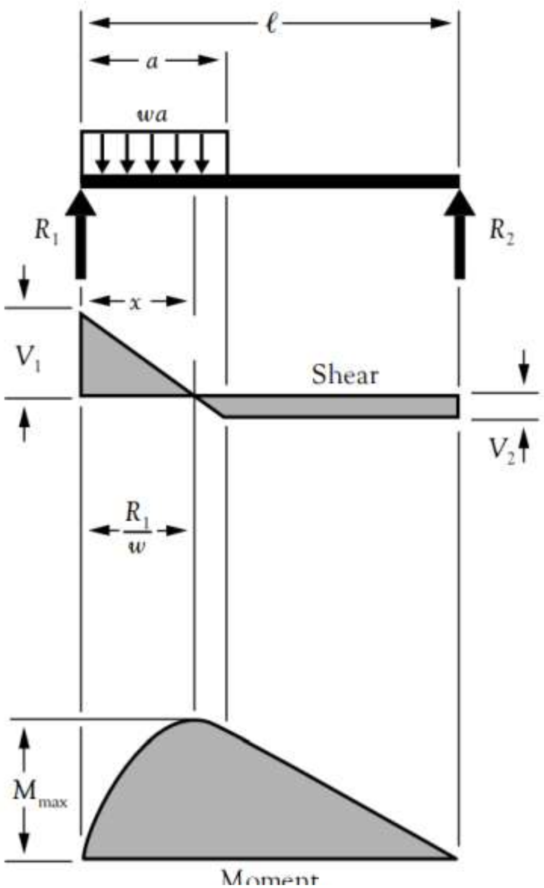
$$M_{max} \left(\text{at } x = a + \frac{R_1}{w}\right) \quad \ldots \quad = R_1\left(a + \frac{R_1}{2w}\right)$$

$$M_x \text{ (when } x < a) \quad \ldots \quad = R_1 x$$

$$M_x \left(\text{when } x > a \text{ and } < (a + b)\right) \quad \ldots \quad = R_1 x - \frac{w}{2}(x - a)^2$$

$$M_x \left(\text{when } x > (a + b)\right) \quad \ldots \quad = R_2(\ell - x)$$

UNIFORMLY DISTRIBUTED PARTIAL LOADING AT ONE END OF SINGLE BEAM



$$R_1 = V_1 \quad \ldots\ldots\ldots\ldots \quad = \frac{wa}{2\ell}(2\ell - a)$$

$$R_2 = V_2 \quad \ldots\ldots\ldots\ldots \quad = \frac{wa^2}{2\ell}$$

$$V_x \text{ (when } x < a) \quad \ldots\ldots \quad = R_1 - wx$$

$$M_{max}\left(\text{at } x = \frac{R_1}{w}\right) \quad \ldots\ldots \quad = \frac{R_1^2}{2w}$$

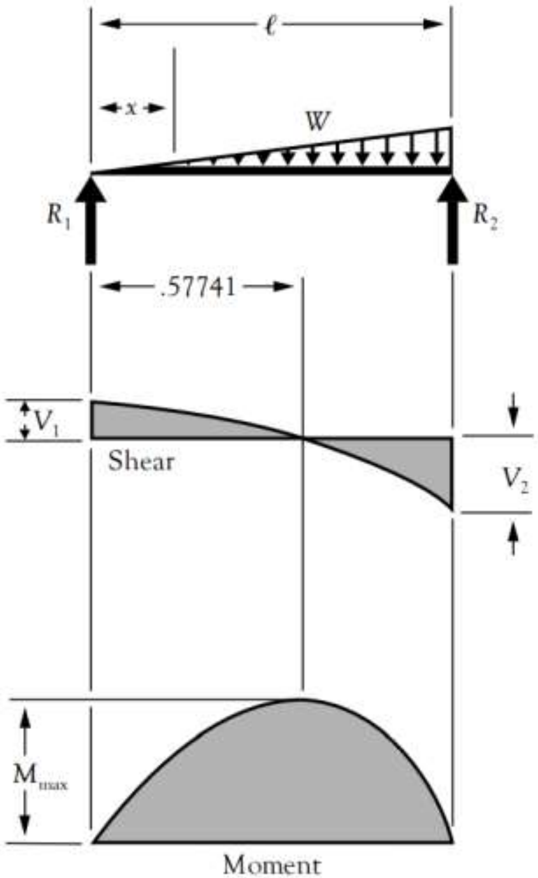
$$M_x \text{ (when } x < a) \quad \ldots\ldots \quad = R_1 x - \frac{wx^2}{2}$$

$$M_x \text{ (when } x > a) \quad \ldots\ldots \quad = R_2(\ell - x)$$

$$\Delta_x \text{ (when } x < a) \quad \ldots\ldots \quad = \frac{wx}{24\,EI\ell}\left(a^2(2\ell - a)^2 - 2ax^2(2\ell - a) + \ell x^3\right)$$

$$\Delta_x \text{ (when } x > a) \quad \ldots\ldots \quad = \frac{wa^2(\ell - x)}{24\,EI\ell}(4x\ell - 2x^2 - a^2)$$

UNIFORMLY INCREASING LOAD TO ONE END



$$R_1 = V_1 \dots\dots = \frac{W}{3}$$

$$R_2 = V_2 \dots\dots = \frac{2W}{3}$$

$$V_x \dots\dots = \frac{W}{3} - \frac{Wx^2}{\ell^2}$$

$$M_{max}\left(\text{at } x = \frac{\ell}{\sqrt{3}} = .5774\ell\right) \dots = \frac{2W\ell}{9\sqrt{3}} = .1283W\ell$$

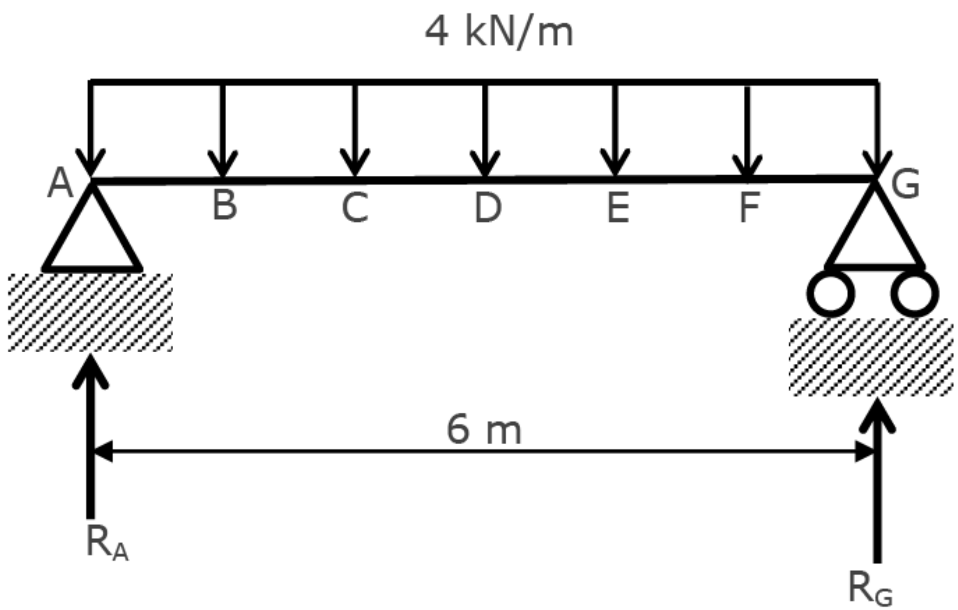
$$M_x \dots\dots = \frac{Wx}{3\ell^2}(\ell^2 - x^2)$$

$$\Delta_{max}\left(\text{at } x = \ell\sqrt{1-\sqrt{\frac{8}{15}}} = .5193\ell\right) \dots = .01304\frac{W\ell^3}{EI}$$

$$\Delta_x \dots\dots = \frac{Wx}{180EI\ell^2}(3x^4 - 10\ell^2x^2 + 7\ell^4)$$

Practical Example 1

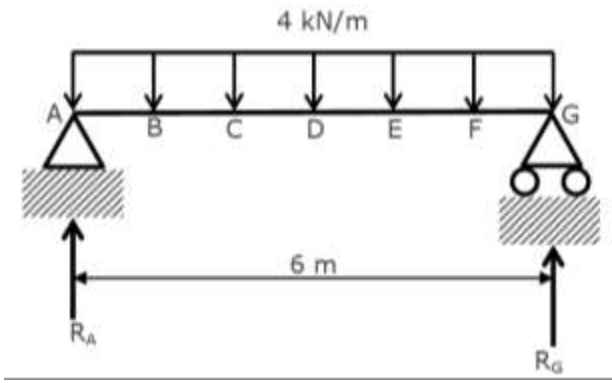
Beam AG, shown in figure 2, spans 6 metres. It supports a uniformly distributed load of 4kN/m along its entire length. Draw the shear force and bending moment diagrams.



$$R_A = R_G = \frac{\left(\frac{4kN}{m} \times 6m\right)}{2} = 12kN$$

SHEAR FORCES

(Remember always look at what's going on to the **left** of the point at which you're trying to calculate shear force.) As before, draw a horizontal straight line representing zero shear force. This will be the base line from which the shear force diagram is drawn.



There is nothing to the left of point A, so the shear force at point A is zero.

If we go a very small distance (say 2 millimetres)to the right of A, there is now a 12 kN upward force (the reaction at A) to the left of the point we're considering. So the shear force at this point is 12 kN. We can represent this effect by a vertical straight line at point A, starting at the zero force base line and going up to a point representing 12 kN.

$$R_A = R_G = \frac{\left(\frac{4kN}{m} \times 6m\right)}{2} = 12kN$$

Each of points B, C, D, E, F and G has this 12 kN upward force to the left of it (i.e. the reaction at point A), but they also have downward forces to the left. Let's consider each of these points in turn.

Point	Upward Force	Downward Force	Shear force
B	12kN	(4kN/m x 1m) =4kN	12 – 4 = 8kN
C	12kN	(4kN/m x 2m) = 8kN	12 – 8 = 4kN
D	12kN	(4kN/m x 3m) = 12kN	12 – 12 = 0kN
E	12kN	(4kN/m x 4m) = 16kN	12 – 16 = – 4kN
F	12kN	(4kN/m x 5m) = 20kN	12 – 20 = – 8kN
G	12kN	(4kN/m x 6m) = 24kN	12 – 24 = – 12kN

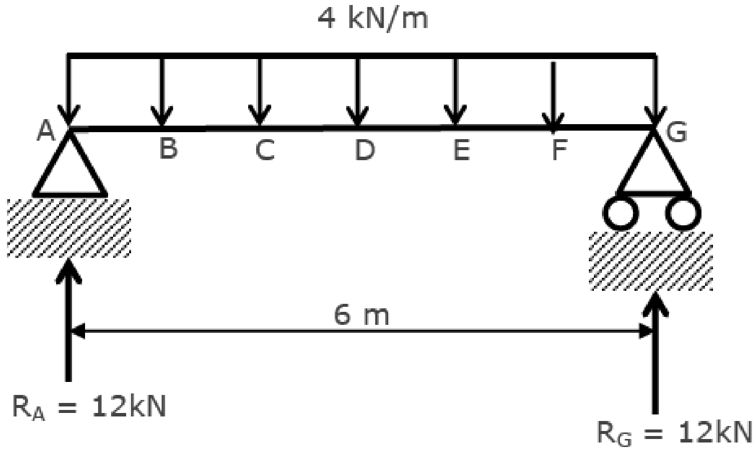
At point G, there is an upward reaction of 12kN. So the net shear force at G will be – 12 + 12 = 0kN.

Bending Moments

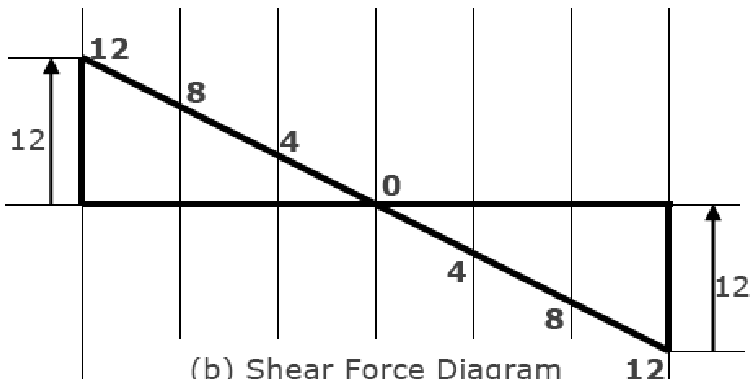
Once more, we will be looking solely at forces and moments to the left of the point we're considering. As in earlier examples, we will calculate the momnet at each point, remembering that:

- Clockwise moments are positive, and anticockwise moments are neagtive;
- Distances are measured from the force concerned to the point considered.

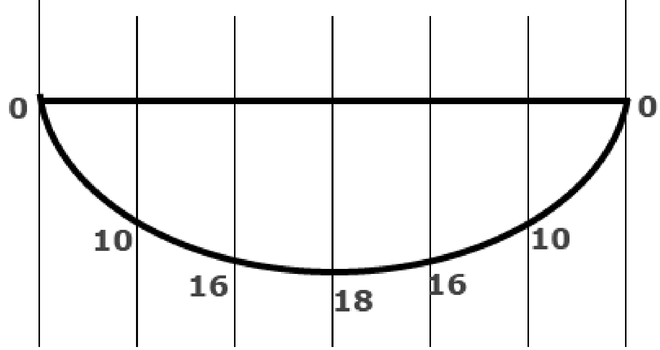
Point	Bending Moment Calculation	Bending Moment (kN.m)
A	+(12kN x 0m)	0
B	+(12kN x 1m) – (4kN/m x 1m x 0.5m)	12 – 2 = 10
C	+(12kN x 2m) – (4kN/m x 2m x 1m)	= 24 – 8 = 16
D	+(12kN x 3m) – (4kN/m x 3m x 1.5m)	= 36 – 18 = 18
E	+(12kN x 4m) – (4kN/m x 4m x 2m)	= 48 – 32 = 16
F	+(12kN x 5m) –(4kN/m x 5m x 2.5m)	= 0 – 50 = 10
G	+(12kN x 6m) – (4kN /m x 6m x 3m)	= 72 – 72 = 0



(a) Beam Diagram

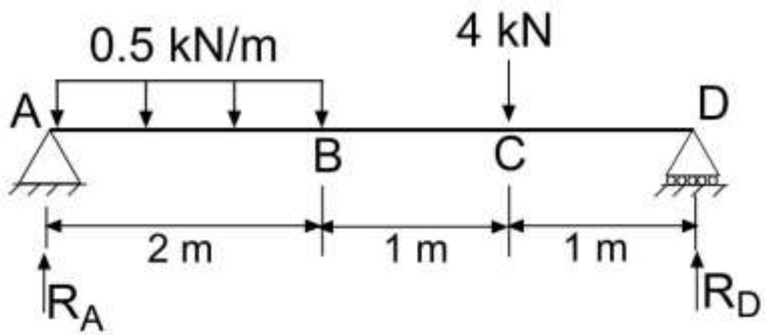


(b) Shear Force Diagram



(c) Bending Moment Diagram

Sample Question : Draw the shear force and bending moment diagram for the single beam

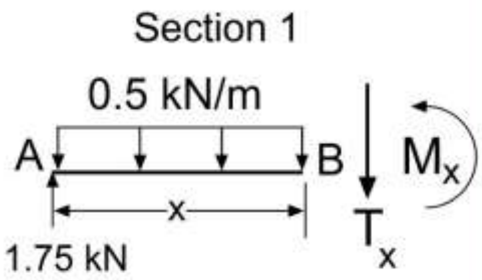


Support reactions

$\sum F_y = 0$; $R_A+R_D - 4 - (0.5x2) = 0$; $R_A+R_D = 5$ kN

$\sum M_A=0$; $4R_D - 12 - 1 = 0$; $R_D=3.25$ kN and $R_A= 1.75$ kN

In order to construct the diagram, the beam is divided into sections based on the location of single and/or distributed loads. The application point of single loads is the zero shear force point where the magnitude of bending moments are maximum. Similarly, the centre of gravity for distributed loads are where the shear forces are zero and bending moment magnitudes are maximum.



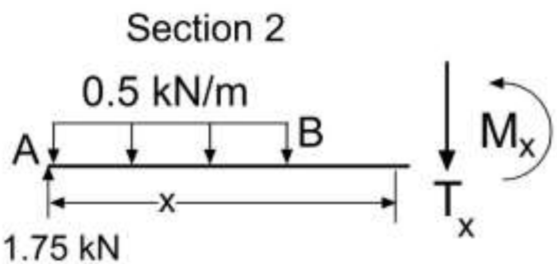
Free-Body Diagram of Section 1:

Due to equilibrium; T_x= 0.5x-1.75

M_{BA}=1.75x-0.5(x^2/2)

Boundary conditions: 0≤x<2 m

At point B (x= 2m); T_x=0.75 kN downwards (+) and M_{BA}=2.5 kN.m CCW



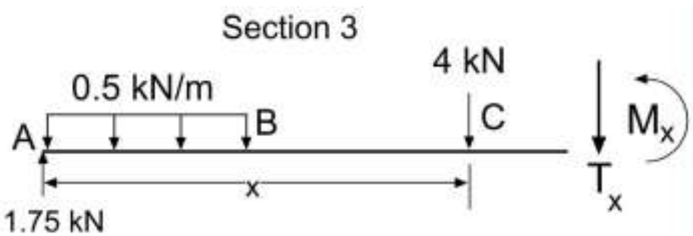
Free-Body Diagram of Section 2:

Due to equilibrium; T_x= -0.75 kN (constant)

M_{CB}=1.75x-1(x-1)

Boundary conditions: 2≤x≤3 m

At point C (x= 3m); T_x=0.75 kN downwards (+) and M_{CB}=3.25 kN.m CCW



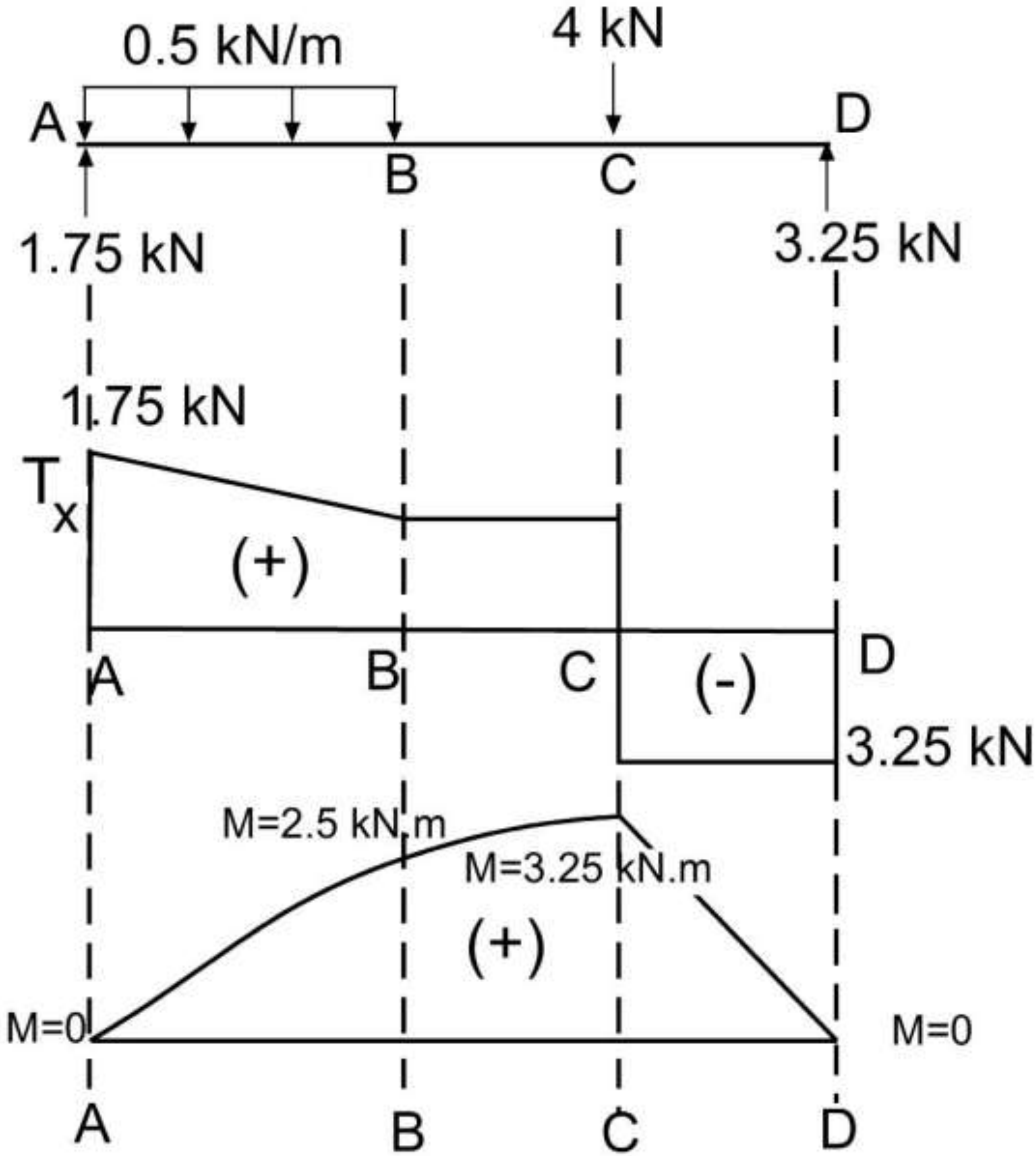
Free-Body Diagram of Section 3:

Due to equilibrium; T_x= 3.25 kN (constant)

M_{CD}=1.75x-1(x-1)-4(x-3)

Boundary conditions: 3≤x≤4 m

At point D (x= 4m); T_x=3.25 kN downwards (+) and M_D=0


0.5 kN/m
4 kN
A
D
B
C
1.75 kN
3.25 kN
1.75 kN
T_x
(+)
A
B
C
(-)
D
3.25 kN
M=2.5 kN.m
M=3.25 kN.m
(+)
M=0
M=0
A
B
C
D

SUMMARY

- Shear forces and moments occur due to loading on beams
- Bending moment is maximum where shearing force is zero
- No bending moment occur where maximum shearing force applied
- It is essential to determine the shear forces along cross sections on a single beam
- Shear forces are indicated by straight lines for point loads, while inclined for distributed loads
- Bending moments and shear forces along beams are graphically represented with distinct sign conventions