Solutions

As was stated before, one of the goals in this course is to solve, or find solutions of differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.

Definition 2:

Any function ϕ defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be **a solution** of the equation on the interval.

EXAMPLE 1(Verification of a Solution)

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $dy/dx = xy^{1/2}$; $\quad y = \frac{1}{16}x^4$ $\qquad$ **(b)** $y'' - 2y' + y = 0$; $\quad y = xe^x$

SOLUTION One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval.

(a) From

left-hand side: $\quad \dfrac{dy}{dx} = \dfrac{1}{16}(4 \cdot x^3) = \dfrac{1}{4}x^3,$

right-hand side: $\quad xy^{1/2} = x \cdot \left(\dfrac{1}{16}x^4\right)^{1/2} = x \cdot \left(\dfrac{1}{4}x^2\right) = \dfrac{1}{4}x^3,$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number x,

left-hand side: $\quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$

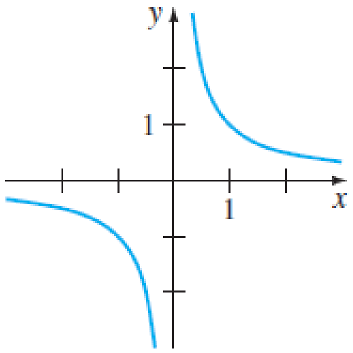
right-hand side: $\quad 0.$ ■

Note, too, that in Example 1 each differential equation possesses the constant solution $y = 0, -\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval *I* is said to be **a trivial solution**.

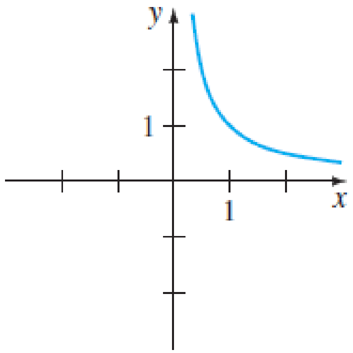
SOLUTION CURVE

The graph of a solution ϕ of an ODE is called a **solution curve**. Since ϕ is a differentiable function, it is continuous on its interval *I* of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ need not be the same as the interval *I* of definition (or domain) of the solution ϕ. Example 2 illustrates the difference.

EXAMPLE 2(Function versus Solution)



(a) function $y = 1/x$, $x \neq 0$



(b) solution $y = 1/x$, $(0, \infty)$

FIGURE 1.1.1 The function $y = 1/x$ is not the same as the solution $y = 1/x$

The domain of $y = 1/x$, considered simply as a *function*, is the set of all real numbers x except 0. When we graph $y = 1/x$, we plot points in the xy-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.1(a). The function $y = 1/x$ is not differentiable at $x = 0$, since the y-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$. (Verify.) But when we say that $y = 1/x$ is a *solution* of this DE, we mean that it is a function defined on an interval I on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0, such as $(-3, -1)$, $\left(\frac{1}{2}, 10\right)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ for $-3 < x < -1$ and $\frac{1}{2} < x < 10$ are simply segments, or pieces, of the solution curves defined by $y = 1/x$ for $-\infty < x < 0$ and $0 < x < \infty$, respectively, it makes sense to take the interval I to be as large as possible. Thus we take I to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ is shown in Figure 1.1.1(b). ■

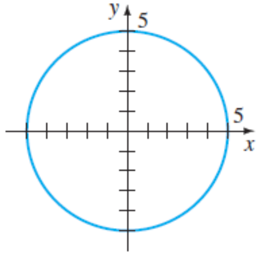
EXPLICIT AND IMPLICIT SOLUTIONS

EXPLICIT AND IMPLICIT SOLUTIONS You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

Definition 3: (Implicit Solution of an ODE)

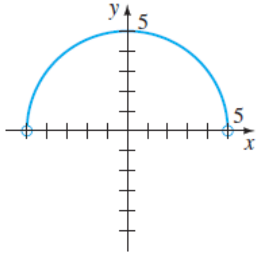
A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation (1) on an interval *I*, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on *I*.

EXAMPLE 3(Verification of an Implicit Solution)



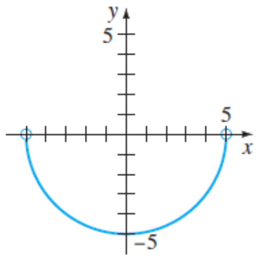
(a) implicit solution

$x^2 + y^2 = 25$



(b) explicit solution

$y_1 = \sqrt{25 - x^2}, \ -5 < x < 5$



(c) explicit solution

$y_2 = -\sqrt{25 - x^2}, \ -5 < x < 5$

FIGURE 1.1.2 An implicit solution and two explicit solutions of $y' = -x/y$

The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{8}$$

on the open interval $(-5, 5)$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0.$$

Solving the last equation for the symbol dy/dx gives (8). Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in Figures 1.1.2(b) and 1.1.2(c) are segments of the graph of the implicit solution in Figure 1.1.2(a). ■

FAMILIES OF SOLUTIONS

The study of differential equations is similar to that of integral calculus. In some texts a solution ϕ is sometimes referred to as **an integral** of the equation, and its graph is called **an integral curve**. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration.

Analogously, when solving a first-order differential equation $F(x,y,y')=0$, we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set $G(x,y,c)=0$ of solutions called a **one-parameter family** of solutions.

When solving an nth-order differential equation $F\left(x,y,y',\ldots,y^{(n)}\right)= 0$, we seek an *n-parameter* family of solutions $G(x,y,c_1,c_2,\ldots,c_n)=0$.

This means that *a single differential equation* can *possess an infinite number of solutions* corresponding to the unlimited number of choices for the parameter(s).

- A solution of a differential equation that *is free of arbitrary parameters* is called a **particular solution.**
- The set of all solutions of a DE is called **general solution.**
- Note that the general solution of a DE involves the same number of parameters with the order of the DE. That is if the DE is 4th order, the parameters in the general solution should be 4.

For example, the one-parameter family

$y = cx - x\cos x$ is an explicit solution of the linear first-order equation $xy' - y = x^2 \sin x$ on the interval $(-\infty, \infty)$. (Verify.) Figure 1.1.3, obtained by using graphing software, shows the graphs of some of the solutions in this family. The solution $y = -x\cos x$, the blue curve in the figure, is a particular solution corresponding to $c = 0$. Similarly, on the interval $(-\infty, \infty)$, $y = c_1e^x + c_2xe^x$ is a two-parameter family of solutions of the linear second-order equation $y'' - 2y' + y = 0$ in Example 1. (Verify.) Some particular solutions of the equation are the trivial solution $y = 0$ $(c_1 = c_2 = 0)$, $y = xe^x$ $(c_1 = 0, c_2 = 1)$, $y = 5e^x - 2xe^x$ $(c_1 = 5, c_2 = -2)$, and so on.

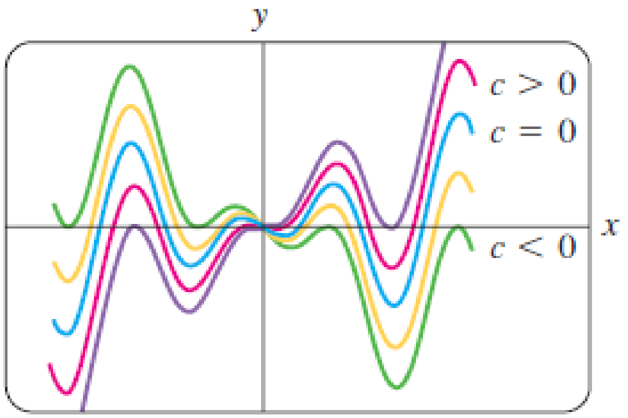


FIGURE 1.1.3 Some solutions of $xy' - y = x^2 \sin x$

Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation —that is, a solution that cannot be obtained by specializing any of the parameters in the family of solutions. Such an extra solution is called a **singular solution**.

For example, we have seen that

$$y = \frac{1}{16}x^4 \text{ and } y = 0$$

are solutions of the differential equation $dy/dx = xy^{1/2}$ on $(-\infty, \infty)$.

1.2 INITIAL-VALUE PROBLEMS

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies prescribed side conditions—that is, conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0 the problem

$$\begin{aligned} &\textit{Solve:} && \frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \qquad (1)$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants, is called an **initial-value problem (IVP).** The values of $y(x)$ and its first $n-1$ derivatives at a single point x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$, are called **initial conditions.**

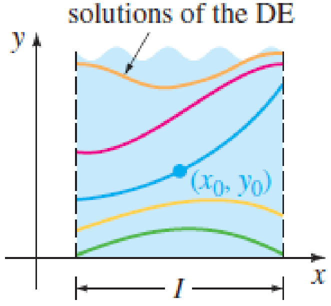


FIGURE 1.2.1 Solution of first-order IVP

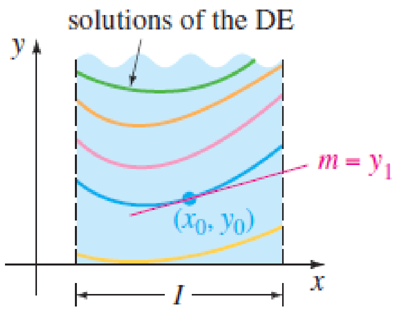


FIGURE 1.2.2 Solution of second-order IVP

FIRST- AND SECOND-ORDER IVPS The problem given in (1) is also called an ***n*th-order initial-value problem.** For example,

$$\text{Solve:} \quad \frac{dy}{dx} = f(x, y) \qquad \text{Subject to:} \quad y(x_0) = y_0 \tag{2}$$

and

$$\text{Solve:} \quad \frac{d^2y}{dx^2} = f(x, y, y') \qquad \text{Subject to:} \quad y(x_0) = y_0,\ y'(x_0) = y_1 \tag{3}$$

are **first-** and **second-order** initial-value problems, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution $y(x)$ of the differential equation $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specified point (x_0, y_0). A solution curve is shown in blue in Figure 1.2.1. For (3) we want to find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval I containing x_0 so that its graph not only passes through (x_0, y_0) but the slope of the curve at this point is the number y_1. A solution curve is shown in blue in Figure 1.2.2. The words *initial conditions* derive from physical systems where the independent variable is time t and where $y(t_0) = y_0$ and $y'(t_0) = y_1$ represent the position and velocity, respectively, of an object at some beginning, or initial, time t_0.

Solving an nth-order initial-value problem such as (1) frequently entails first finding an n-parameter family of solutions of the given differential equation and then using the n initial conditions at x_0 to determine numerical values of the n constants in the family. The resulting particular solution is defined on some interval I containing the initial point x_0.

Solve questions.

Existence and Uniqueness

Two fundamental questions arise in considering an initial-value problem:

- Does a solution of the problem exist?
- If a solution exists, is it unique?

Example 4 (An IVP Can Have Several Solutions)

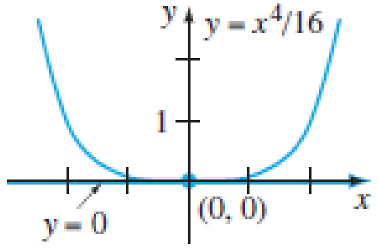


FIGURE 1.2.5 Two solutions of the same IVP

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions pass through the same point (0, 0). ■

Theorem 1 (Existence of a Unique Solution)

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

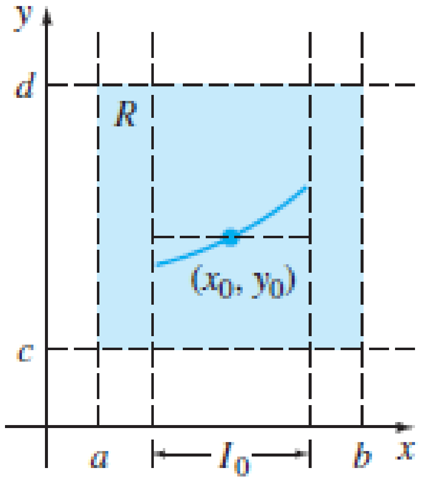


FIGURE 1.2.6 Rectangular region R

Example 5 (Revisited Example 4)

We saw in Example 4 that the differential equation

$$\frac{dy}{dx} = xy^{1/2}$$

possesses at least two solutions whose graphs pass through $(0, 0)$. Inspection of the functions

$$f(x, y) = xy^{1/2} \quad and \ \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane defined by $y > 0$. Hence Theorem 1 enables us to conclude that through any point (x_0, y_0), $y_0 > 0$ in the upper half-plane there is some interval centered at x_0 on which the given differential equation has a unique solution.

Thus, for example, even without solving it, we know that there exists some interval centered at 2 on which the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, y(2) = 1$$

has a unique solution.

Remark

The conditions in Theorem 1 are sufficient but not necessary. This means that when $f\,(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on a rectangular region R, it must always follow that a solution of (2) exists and is unique whenever (x_0, y_0), is a point interior to R.

However, if the conditions stated in the hypothesis of Theorem 1 do not hold, then anything could happen:

Problem (2) ,

- may still have a solution and this solution may be unique,

or

- (2) may have several solutions,

or

- it may have no solution at all.