11. LAPLACE TRANSFORMS

11.1. INTRODUCTION:

From calculus, we know that differentiation and integration are **transforms;** this means that these operations transform a function into another function.

For example, the function $f\,(x) = x^2$ is transformed into a linear function and a family of cubic polynomial functions by the operations of differentiation and integration:

$$\frac{d}{dx}x^2 = 2x \qquad \text{and} \qquad \int x^2\,dx = \frac{1}{3}x^3 + c.$$

- Moreover these two transforms are linear, i.e.

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

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In this section we will examine a special type of integral transform called the **Laplace transform**. In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear initial-value problems.

- Laplace transform is a special type of transform, which transforms a suitable function f of a real variable t into a related function F of a real variable s.
- **The most important point of this transforms is: Laplace transform tranforms the IVP consisting of DEs to an Algebric Equation**

INTEGRAL TRANSFORM If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, by holding y constant, we see that $\int_1^2 2xy^2\,dx = 3y^2$. Similarly, a definite integral such as $\int_a^b K(s, t)\,f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an **integral transform,** where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s, t)\,f(t)\,dt$ is defined as a limit:

$$\int_0^\infty K(s, t)\,f(t)\,dt = \lim_{b\to\infty} \int_0^b K(s, t)\,f(t)\,dt. \tag{1}$$

If the limit in (1) exists, then we say that the integral exists or is **convergent;** if the limit does not exist, the integral does not exist and is **divergent.** The limit in (1) will, in general, exist for only certain values of the variable s.

A DEFINITION The function $K(s, t)$ in (1) is called the **kernel** of the transform. The choice $K(s, t) = e^{-st}$ as the kernel gives us an especially important integral transform.

Definition (Laplace Transform)

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt \qquad (2)$$

is said to be the Laplace transform of f, provided that the integral converges.

When the defining integral (2) converges, the result is a function of s. In general discussion we shall use a **lowercase letter** to denote the function *being transformed* and the corresponding **capital letter** to denote its *Laplace transform—for* example,

$$\mathcal{L}\{f(t)\} = F(s), \qquad \mathcal{L}\{g(t)\} = G(s), \qquad \mathcal{L}\{y(t)\} = Y(s),$$

Example

Evaluate $\mathscr{L}\{1\}$.

SOLUTION From (2),

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty}\frac{-e^{-st}}{s}\Big|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

provided that $s > 0$. In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$. ■

The use of the limit sign becomes somewhat tedious, so we shall adopt the notation $|_0^\infty$ as a shorthand for writing $\lim_{b\to\infty}(\ )\,|_0^b$. For example,

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \frac{-e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}, \qquad s > 0.$$

At the upper limit, it is understood that we mean $e^{-st} \to 0$ as $t \to \infty$ for $s > 0$.

Example: Evaluate the following Laplace transforms

1. $\mathcal{L}\{t\}$
2. $\mathcal{L}\{e^{-3t}\}$
3. $\mathcal{L}\{sin2t\}$

$\mathscr{L}$ IS A LINEAR TRANSFORM For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\,dt = \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s). \tag{3}$$

Because of the property given in (3), $\mathscr{L}$ is said to be a **linear transform.** For example, from Examples 1 and 2

$$\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2},$$

and from Examples 3 and 4

$$\mathscr{L}\{4e^{-3t} - 10 \sin 2t\} = 4\mathscr{L}\{e^{-3t}\} - 10\mathscr{L}\{\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}.$$

THEOREM (Transforms of Some Basic Functions)

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s - a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

THEOREM (Sufficient Conditions for Existence)

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

A function f is said to be of **exponential order** c if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Example

Evaluate $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3. \end{cases}$

SOLUTION The function f, shown in Figure 7.1.5, is piecewise continuous and of exponential order for $t > 0$. Since f is defined in two pieces, $\mathscr{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt$$

$$= 0 + \left.\frac{2e^{-st}}{-s}\right|_3^\infty$$

$$= \frac{2e^{-3s}}{s}, \qquad s > 0.$$

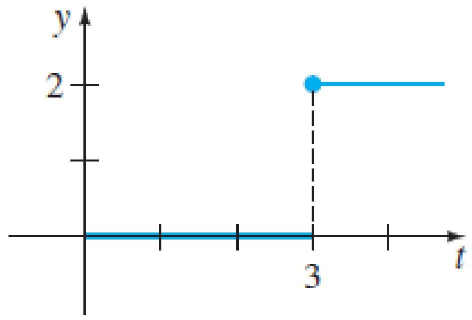

FIGURE 7.1.5 Piecewise continuous function

11.2. PROPERTIES OF THE LAPLACE TRANSFORM

In the previous section, we defined the Laplace transform of a function $f(t)$ as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Using this definition to get an explicit expression for $\mathcal{L}\{f(t)\}$ requires the evaluation of the improper integral—frequently **a tedious task**! We have already seen how the linearity property of the transform can help relieve this burden.

In this section we discuss some further properties of the Laplace transform that simplify its computation. These new properties will also enable us to use the Laplace transform to solve initial value problems.

Assume that $F(s)$ is a Laplace Transform of the function $f(t)$.

Now we can give the following properties:

1. $\mathcal{L}\{f(x)e^{ax}\} = F(s-a)$
2. $\mathcal{L}\{f(x)x^n\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Solve Questions