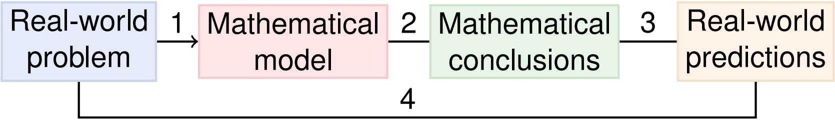
Mathematical Models

A **mathematical model** is a mathematical description of a real-world phenomenon.



1. Formulate
   Identify independent & dependent variables, simplify and obtain equations (possibly guessing from measurements).
2. Solve
   Apply mathematics such as calculus to derive conclusions.
3. Interpret
   Interpret the model conclusions to predict the real-world.
4. Test
   Compare predictions with reality (revise model if needed).

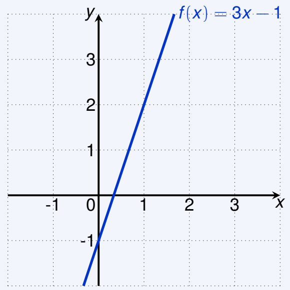
Linear Functions

A **linear function** is a function f that can be written in the form:

$$f(x) = mx + b$$

where m is the **slope** and b is the y**-intercept.**

The graph of a linear function is a line:



Linear Functions: Example

When dry air moves upward it expands and cools.

- ground temperature is 20°
- temperature in height of 1km is 10°

Express the temperature as a linear function of the height *h*.
What is the temperature in 2.5km height?

Since we are looking for a linear function:

$$T(h) = mh + b$$

We know that:

$$T(0) = m \cdot 0 + b = 20 \quad \Longrightarrow \quad b = 20$$
$$T(1) = m \cdot 1 + b = m \cdot 1 + 20 = 10 \quad \Longrightarrow \quad m = 10 - 20 = 10$$

Thus $T(h) = -10m + 20$, and $T(2.5) = -5^\circ$.

Polynomials

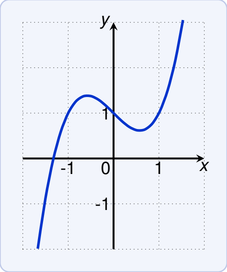
A function P is called **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

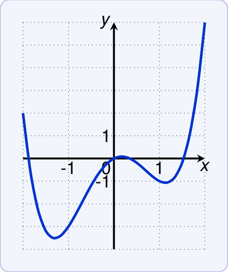
where

- n is a non-negative integer, and
- $a_0, a_1, \ldots, a_n$ are constants, called **coefficients.**

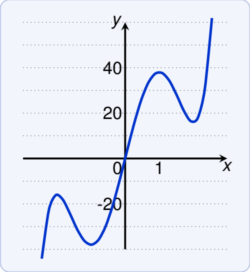
If $a_n \neq 0$ then n is the **degree** of the polynomial.



$x^3 - x + 1$



$x^4 - 3x^2 + x$

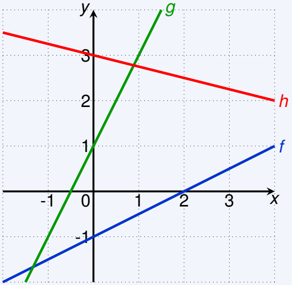


$3x^5 - 25x^3 + 60x$

Polynomials of Degree 1: Linear Functions

A polynomial of degree 1 is a **linear function**:

$$f(x) = mx + b$$



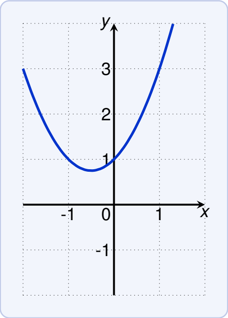
Find equations for the functions f, g and h:

- for f: $f(x) = \frac{1}{2}x - 1$
- for g: $f(x) = 2x + 1$
- for h: $f(x) = -\frac{1}{4}x + 3$

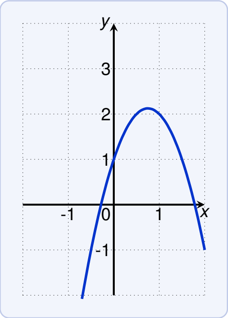
Polynomials of Degree 2: Quadratic Functions

A polynomial of degree 2 is a **quadratic function**:

$$f(x) = ax^2 + bx + c$$



$x^2 + x + 1$



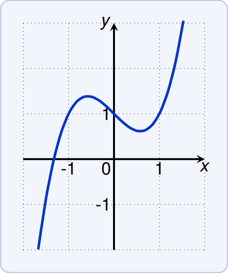
$-2x^2 + 3x + 1$

The graph of is always a shifting of the parabola ax^2. It open upwards if $a > 0$, and downwards if $a < 0$.

Polynomials of Degree 3: Cubic Functions

A polynomial of degree 3 is a **cubic function**:

$$f(x) = ax^3 + bx^2 + cx + d$$



$x^3 - x + 1$

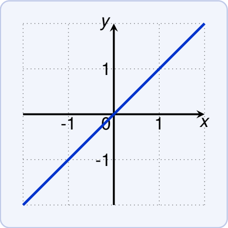
Power Functions

A function of the form

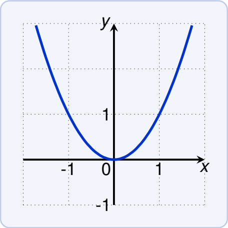
$$f(x) = x^a$$

where a is a constant, is called a **power function.**

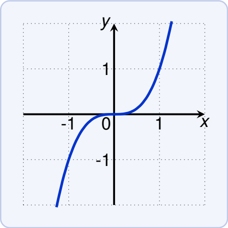
x^1



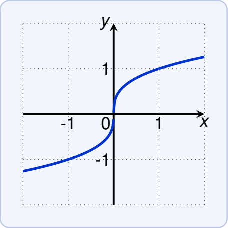
x^2



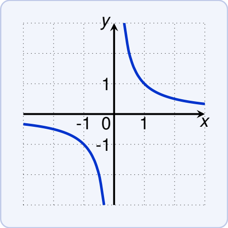
x^3



$x^{\frac{1}{3}}$



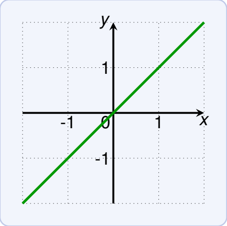
x^{-1}



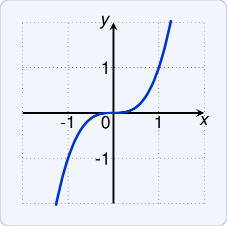
Power Functions: Special Cases

We consider x^n with n a positive integer.

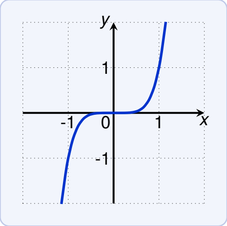
x^1



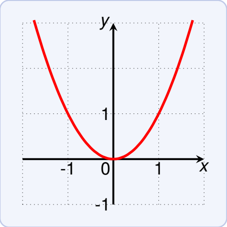
x^3



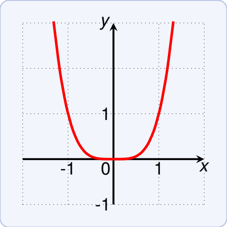
x^5



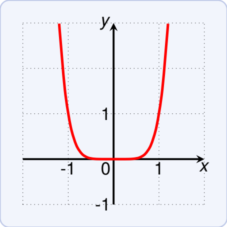
x^2



x^4



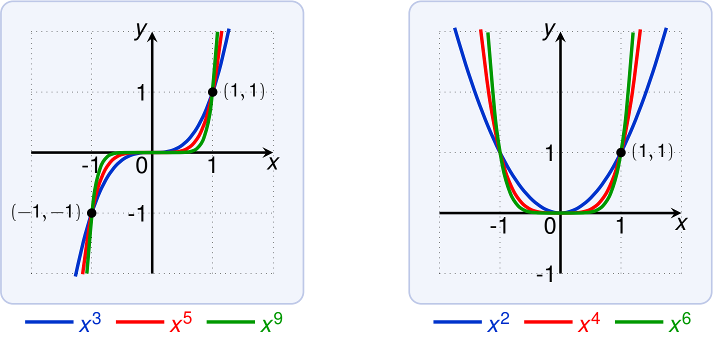
x^6



Power Functions: Special Cases

We consider x^n with n a positive integer.

- For even n the graph similar to the parabola x^2.
- For odd n the graph looks similar to x^3.

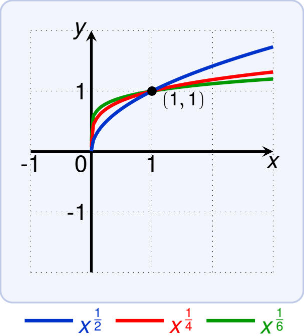


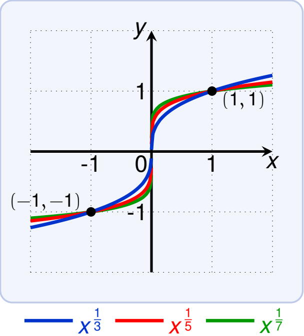
If n increases, then the graph of x^n becomes flatter near 0, and steeper for $|x| \geq 1$.

Power Functions: Special Cases

We consider $x^{\frac{1}{n}}$ where n is a positive integer:

- $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a **root function** (square root for $n = 2$)

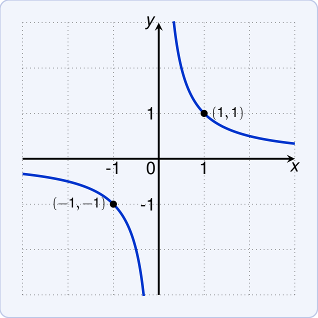


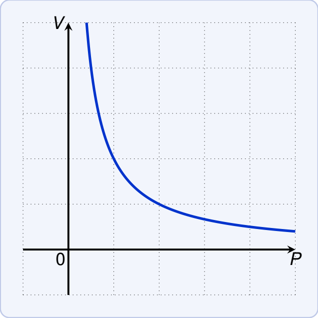


- For even n the domain is $[0, \infty)$, the graph is similar to $\sqrt{x}$.
- For odd n the domain is $\mathbb{R}$, the graph is similar to $\sqrt[3]{x}$.

Power Functions: Special Cases

The power function $f(x) = x^{-1} = \frac{1}{x}$ is the **reciprocal function.**





This function arises in physics and chemistry. E.g. Boyle's law says that, when the temperature is constant, then the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P} \qquad \text{where } C \text{ is a constant}$$

Power Function: Applications

Power functions are used for modeling:

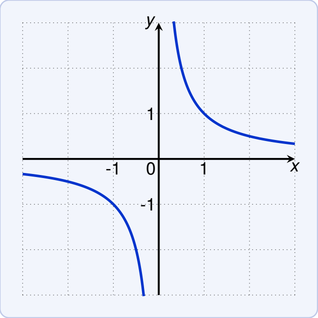
- the illumination as a function of the distance from a light source
- the period of the revolution of a planet as a function of the distance from the sun

Rational Functions

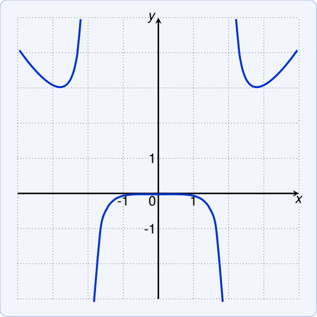
A **rational function** f is ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)} \qquad \text{where } P \text{ and } Q \text{ are polynomials}$$

- the domain of $\frac{P(x)}{Q(x)}$ is $\{\, x \mid Q(x) \neq 0 \,\}$



$f(x) = \frac{1}{x}$

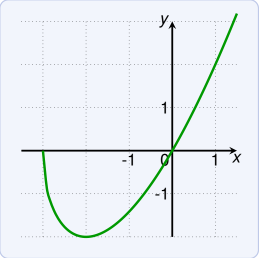


$f(x) = \frac{2x^4 - x^2 + 1}{10x^2 - 40}$

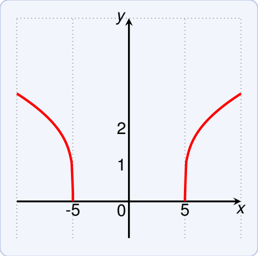
Algebraic Functions

A function f is called **algebraic function** if it can be constructed using algebraic operations (addition, subtraction, multiplication, division and taking roots) starting with polynomials.

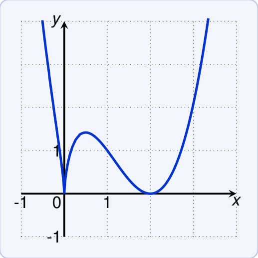
$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^2 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$$



$x\sqrt{x+3}$



$\sqrt[4]{x^2 - 25}$



$x^{\frac{2}{3}}(x-2)^2$

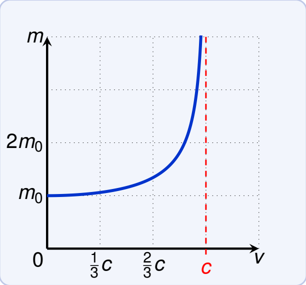
Algebraic Functions: Real-wold Example

The following algebraic function occurs in the theory of relativity. The mass of an object with velocity v is:

$$m = f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where

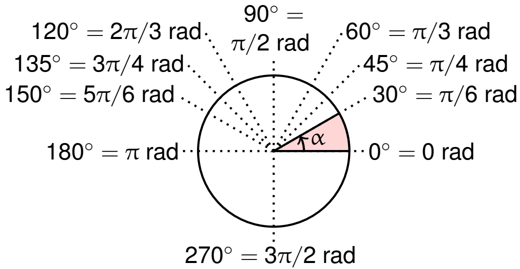
- m_0 is the rest mass of the object
- $c \approx 3.0 \cdot 10^5 \frac{\text{km}}{\text{h}}$ is the speed of light (in vacuum)



Angles

Angles can be measured in **degrees** ($^\circ$) or in **radians** (rad):

- $180^\circ = \pi$ rad
- $360^\circ = 2\pi$ rad is a full revolution



From $180^\circ = \pi$ rad we conclude

$$1^\circ = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad x^\circ = \frac{x \cdot \pi}{180} \text{ rad}$$

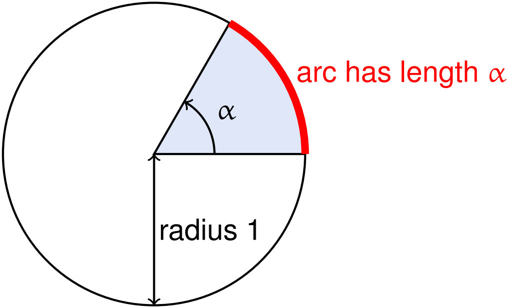
$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \qquad \text{and} \qquad x \text{ rad} = \left(\frac{x \cdot 180}{\pi}\right)^\circ$$

Angles: Radian

In Calculus, the default measurement for angles is **radian.**

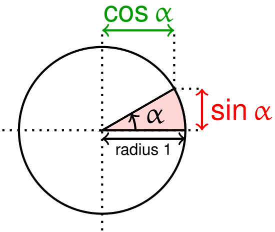
Historical note on radians:

- consider a circle with radius 1, and
- an sector of this circle with angle α (radians)



Then the arc of the sector has length α (equal to the angle).

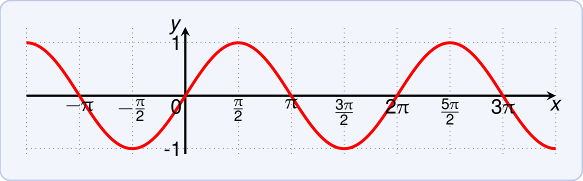
Trigonometric Functions



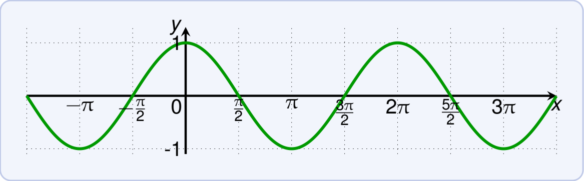
Properties of sin and cos:

- domain = ?
- range = ?

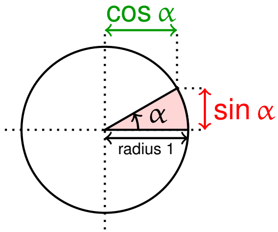
sin x



cos x



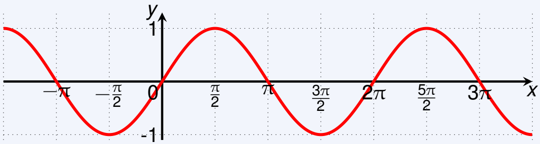
Trigonometric Functions



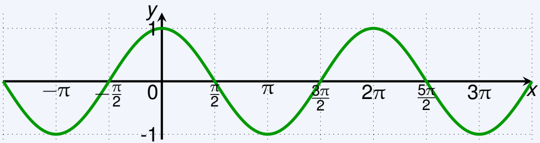
Properties of sin and cos:

- domain = $(-\infty, \infty)$
- range = $[-1, 1]$

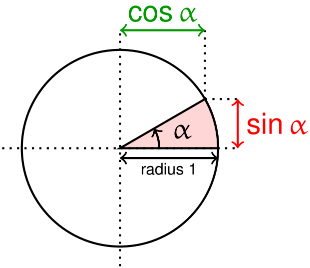
sin x



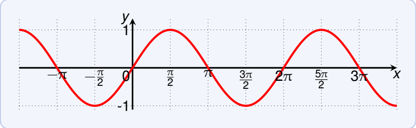
cos x



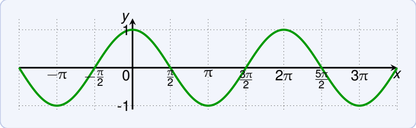
Trigonometric Functions: Identities



sin x

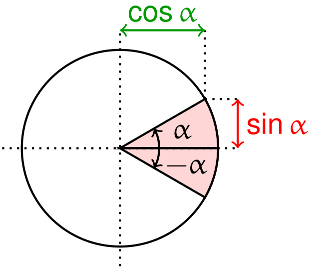


cos x

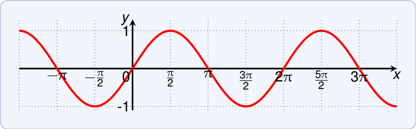


Important identities:

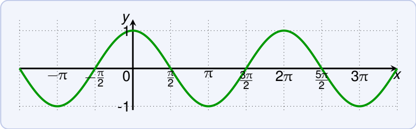
Trigonometric Functions: Identities



sin x



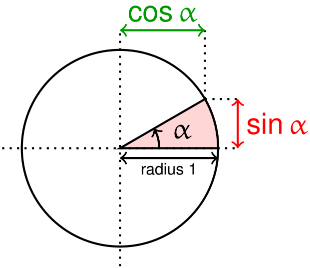
cos x



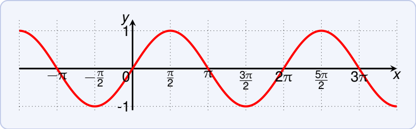
Important identities:

- $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$

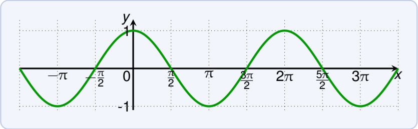
Trigonometric Functions: Identities



sin x



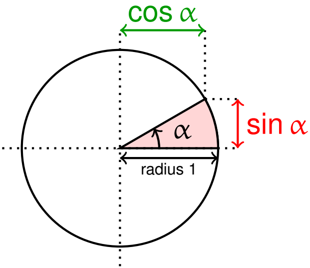
cos x



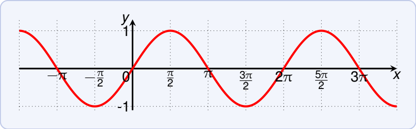
Important identities:

- $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$
- $\sin(\alpha + 2\pi) = \sin\alpha$ and $\cos(\alpha + 2\pi) = \cos\alpha$
- $\cos\alpha = \sin(\alpha \pm ?)$

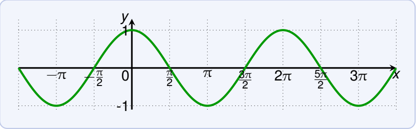
Trigonometric Functions: Identities



sin x



cos x



Important identities:

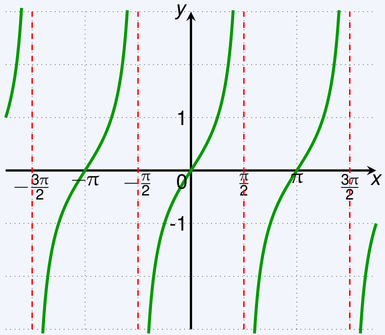
- $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$
- $\sin(\alpha + 2\pi) = \sin\alpha$ and $\cos(\alpha + 2\pi) = \cos\alpha$
- $\cos\alpha = \sin(\alpha + \frac{\pi}{2})$
- $\sin^2\alpha + \cos^2\alpha = 1$ (follows form the Pythagorean theorem)

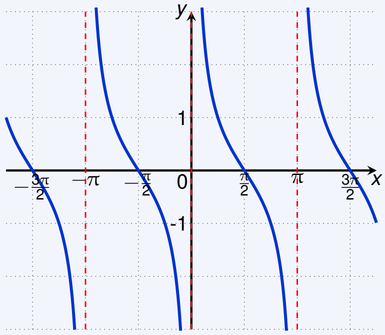
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Trigonometric Functions: Tangent and Cotangent

The tangent and cotangent are defined as:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$





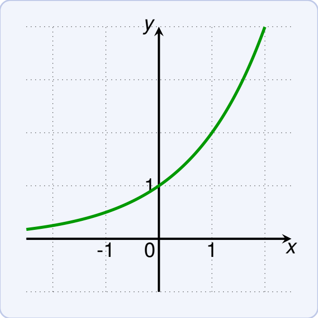
- range = $(-\infty, \infty)$
- domain of tan = $\{x \mid \cos x \neq 0\} = \mathbb{R} \setminus \{\pi/2 + z\pi \mid z \in \mathbb{Z}\}$
- domain of cot = $\{x \mid \sin x \neq 0\} = \mathbb{R} \setminus \{z\pi \mid z \in \mathbb{Z}\}$

Exponential Functions

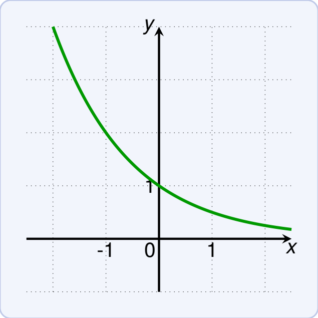
An **exponential function** is a function of the form

$$f(x) = a^x$$

where the **base** a is positive real number ($a > 0$).



$f(x) = 2^x$



$f(x) = 0.5^x$

These functions are called exponential since the variable x is in the exponent. Do not confuse them with power functions x^a!

Exponential Functions

How is a^x defined for $x \in \mathbb{R}$?

For $x = 0$ we have $a^0 = 1$.

For positive integers $x = n \in \mathbb{N}$ we have

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n\text{-times}}$$

For negative integers $x = -n$ we have

$$a^{-n} = \frac{1}{a^n}$$

For rational numbers $x = \frac{p}{q}$ with p, q integers we have

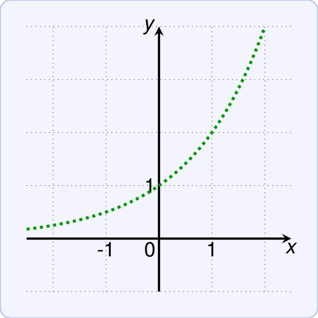
$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

$$4^{\frac{3}{2}} = (\sqrt[2]{4})^3 = 2^3 = 8$$

Exponential Functions: Irrational Numbers

But what about irrational numbers? What is $2^{\sqrt{3}}$ or 5^{π}?

Roughly, one can imagine the situation like in this figure:



We have have defined the function for all rational points, and now want to close the gaps.

Clearly, the result should be an increasing function. . .

Exponential Functions: Irrational Numbers

But what about irrational numbers? What is $2^{\sqrt{3}}$ or 5^{π}?

By increasingness we know:

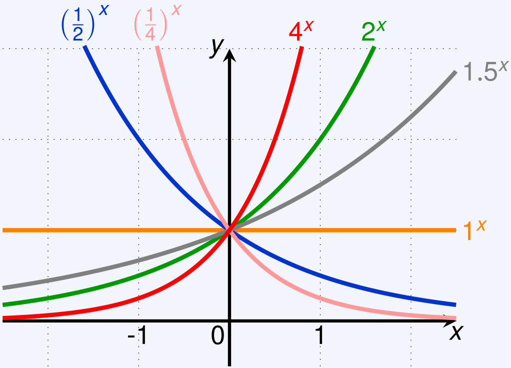
$$1.73 < \sqrt{3} < 1.74 \implies 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$$
$$1.732 < \sqrt{3} < 1.733 \implies 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$$
$$1.7320 < \sqrt{3} < 1.7321 \implies 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$$
$$1.73205 < \sqrt{3} < 1.73206 \implies 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}$$
$$\vdots$$

There is exactly one number that fulfills all conditions on the right.

E.g., $2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}$ determines the first 6 digits:

$$2^{\sqrt{3}} \approx 3.321997$$

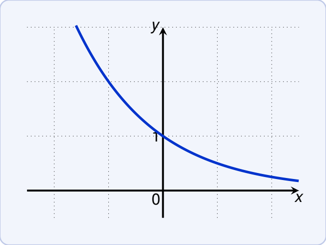
Exponential Functions: Examples



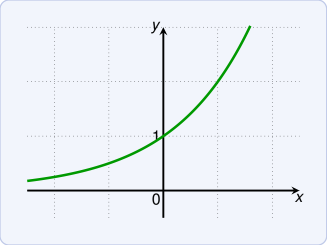
Properties:

- All exponential functions pass through $(0, 1)$ (since $a^0 = 1$)
- Larger base a yields more rapid growth for $x > 0$.

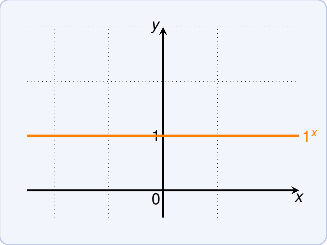
Exponential Functions: Three Types



$f(x) = a^x$ with $0 < a < 1$



$f(x) = a^x$ with $a > 1$



$f(x) = 1^x$

- constant for $a = 1$
- increasing for $a > 1$
- decreasing for $0 < a < 1$
- domain = $(-\infty, \infty)$
- range = $(0, \infty)$ if $a \neq 1$

Laws of Exponents

Laws of Exponents

If a and b are positive real numbers, then:

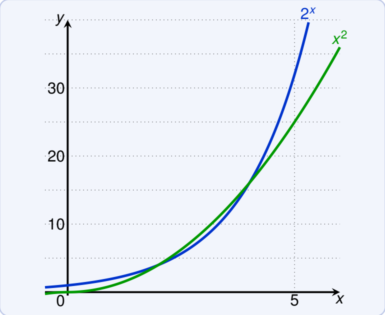
1. $a^{x+y} = a^x \cdot a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

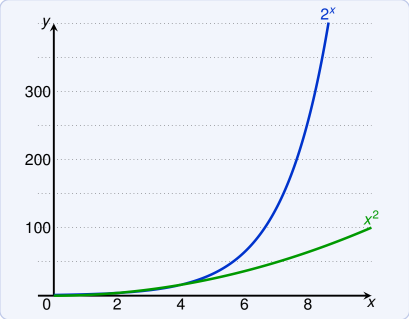
1. $a^{3+4} = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a) = a^3 \cdot a^4$
2. $a^{5-2} = a \cdot a \cdot a = \frac{(a \cdot a \cdot a) \cdot (a \cdot a)}{a \cdot a} = \frac{a^5}{a^2}$
3. $(a^2)^3 = (a \cdot a)^3 = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a^6 = a^{2 \cdot 3}$
4. $(ab)^3 = (ab) \cdot (ab) \cdot (ab) = (a \cdot a \cdot a) \cdot (b \cdot b \cdot b) = a^3 b^3$

Exponential Functions vs. Power Functions

Which functions grows quicker when x is large:

$$f(x) = x^2 \qquad\qquad g(x) = 2^x$$



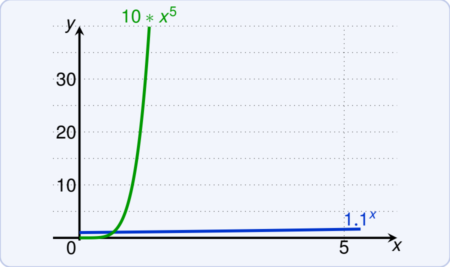


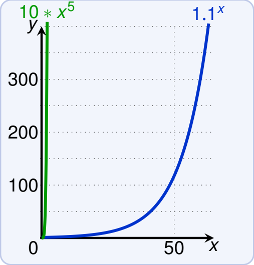
For large x, the function 2^x grows much much faster than x^2.

Exponential Functions vs. Power Functions

Which functions grows quicker when x is large:

$$f(x) = 10 \cdot x^5 \qquad\qquad g(x) = 1.1^x$$





Exponential Functions vs. Power Functions

Which functions grows quicker when x is large:

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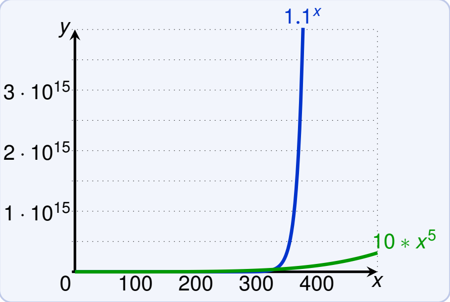


For any $1 < a$, the **exponential function** $f(x) = a^x$ grows for large x much **faster than any polynomial.**

Exponential Functions: Applications

We consider a population of bacteria:

- suppose the population doubles every hour
- we write $p(t)$ for the population after t hours
- initial population is $p(0) = 1000$

We have:

$$
\begin{aligned}
p(1) &= 2 \cdot p(0) = 2 \cdot 1000 \\
p(2) &= 2 \cdot p(1) = 2^2 \cdot 1000 \\
p(3) &= 2 \cdot p(2) = 2^3 \cdot 1000 \\
&\vdots
\end{aligned}
$$

Thus in general

$$p(t) = 1000 \cdot 2^t$$

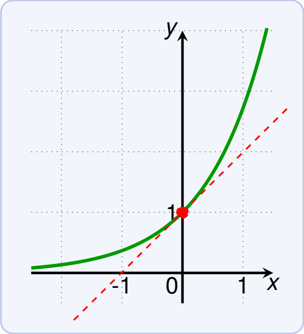
Under ideal conditions such rapid growth occurs in nature.

Exponential Functions: The Number e

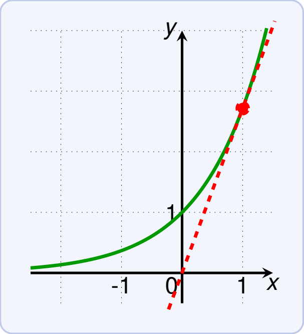
The number

$$e \approx 2.71828\ldots$$

is a very special base for exponential functions.



tangent has slope $1 = e^0$



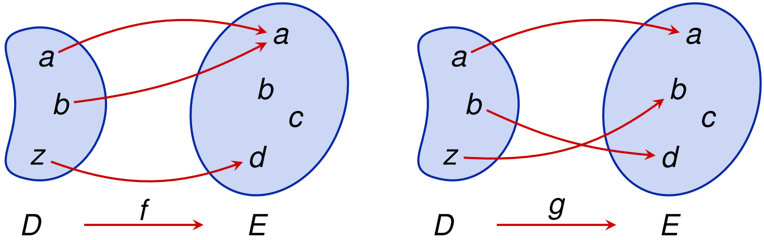
tangent has slope $e = e^1$

The slope of the function e^x at point (x, e^x) is e^x.

One-To-One Functions

A **one-to-one function** is a function that never takes the same value twice, that is:

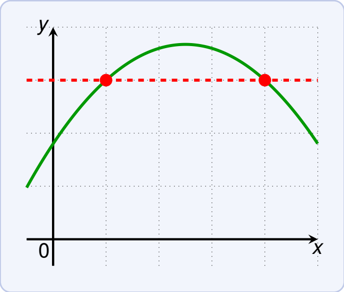
$$f(x) \neq f(y) \quad \text{whenever } x \neq y$$



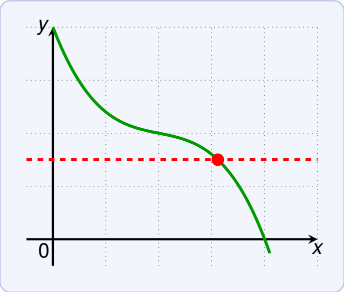
Which of these function is one-to-one? The function g.

One-To-One Functions

How can we see from a graph if the function is one-to-one?



not one-to-one



one-to-one

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

One-To-One Functions: Examples

Which of the following functions is one-to-one?

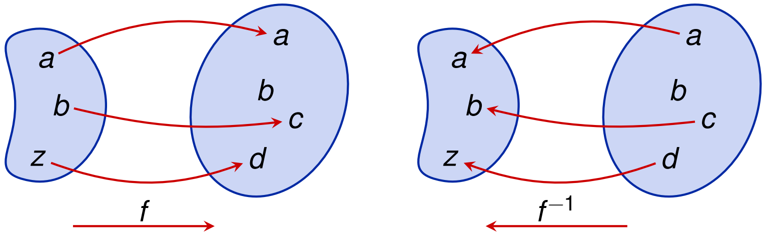
- x^3 ? Yes
- x^2 ? No
- 4^x ? Yes
- $x - x^3$? No
- $x + 4^x$? Yes
- $-x - x^3$? Yes

Inverse Functions

A function g is the inverse of a function f if

$$g(f(x)) = x \quad \text{for all } x \text{ in the domain of } f$$

(and the domain of g is the range of f).



A function f has an inverse if and only if f is one-to-one.

Inverse Functions

The inverse of a one-to-one function can be defined as follows.

Let f be a one-to-one function with domain A and range B.

Then its **inverse function** f^{-1} is defined by:

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y$$

and has domain B and range A.

The inverse function of $f(x) = x^3$ is $f^{-1}(y) = y^{\frac{1}{3}}$:

$$f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$$

We have the following **cancellation equations**:

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in A$$

$$f(f^{-1}(y)) = y \qquad \text{for all } y \in B$$

Inverse Functions

To find the inverse function of f:

- solve the equation $y = f(x)$ for x in terms of y

Find the inverse function of $f(x) = x^3 + 2$.

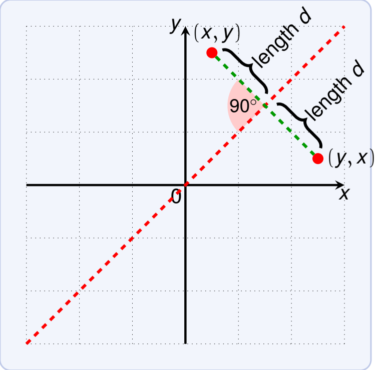
$$\begin{aligned} y &= x^3 + 2 \\ \implies x^3 &= y - 2 \\ \implies x &= \sqrt[3]{y-2} \end{aligned}$$

Therefore the inverse function of f is $f^{-1}(y) = \sqrt[3]{y-2}$

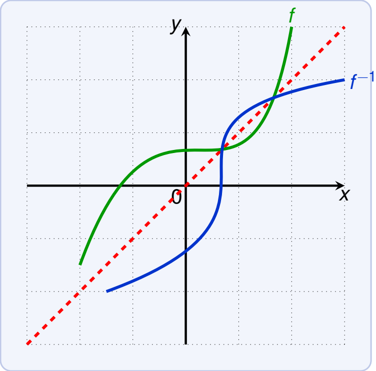
Inverse Functions: Graphs

We have $f(x) = y \iff f^{-1}(y) = x$ and hence

$$\text{point } (x, y) \text{ in the graph of } f \iff \text{point } (y, x) \text{ in the graph of } f^{-1}$$



reflected about the line $y = x$



Logarithmic Functions

The **logarithmic functions**

$$f(x) = \log_a x$$

where $a > 0$ and $a \neq 1$.

The function $\log_a x$ is the inverse of the exponential function a^x:

$$\log_a y = x \iff a^x = y$$

The logarithm $\log_a b$ gives us the exponent for a to get b.

For example: $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$.

The logarithmic functions $\log_a x$ have:

- domain = $(0, \infty)$
- range = $\mathbb{R}$

Logarithmic Functions

We have the following cancellation equations:

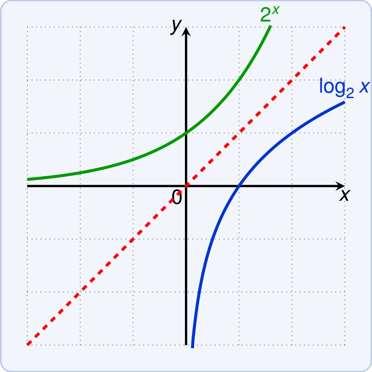
$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

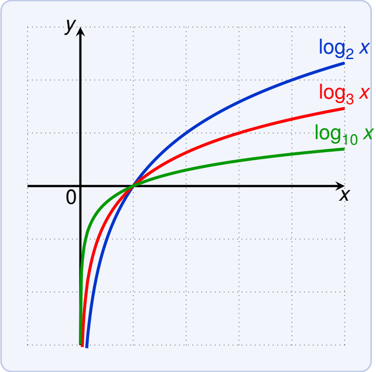
$$a^{\log_a x} = x \quad \text{for every } x > 0$$

$$\log_{10}(10^{23}) = 23$$

$$5^{log_5 7} = 7$$

Logarithmic Functions





For $a > 1$, $f(x) = a^x$ grows very fast.

As a consequence:

For $a > 1$, $f(x) = \log_a x$ grows very slow.

Logarithmic Functions: Laws of Logarithm

If $x, y > 0$, then

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
3. $\log_a(x^r) = r \log_a x$

$$\log_2 80 - \log_2 5 = \log_2(\frac{80}{5}) = \log_2 16 = 4$$

We can proof the laws from the laws for exponents.

1. $\log_a(xy) = z \iff a^z = xy$

   and $a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)} \cdot a^{\log_a(y)} = xy$

3. $\log_a(x^r) = z \iff a^z = x^r$

   and $a^{r \log_a(x)} = (a^{\log_a(x)})^r = x^r$

Logarithmic Functions: Base Conversion

If we want to compute $\log_a x$ but have only $\log_b$ then we can:

Base Conversion

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Compute $\log_4 16$ using $\log_2$.

$$\log_4 16 = \frac{\log_2 16}{\log_2 4} = \frac{4}{2} = 2$$

Natural Logarithm

The **natural logarithm** ln is a special logarithm with base e:

$$\ln x = \log_e x$$

Solve the equation $e^{5-3x} = 10$.

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 && \text{apply natural logarithm on both sides} \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \frac{5 - \ln 10}{3} \end{aligned}$$

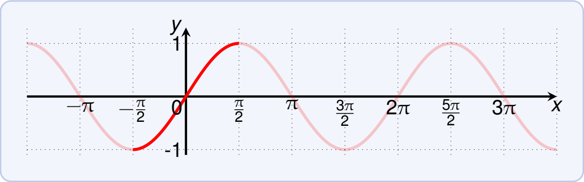
Express $\ln a + \frac{1}{2}\ln b$ in a single logarithm.

$$\ln a + \frac{1}{2}\ln b = \ln a + \ln b^{\frac{1}{2}} = \ln a + \ln\sqrt{b} = \ln(a\sqrt{b})$$

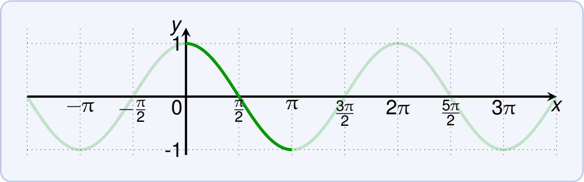
Inverse Trigonometric Functions

We are interested in inverse functions of:

sin *x*



cos *x*

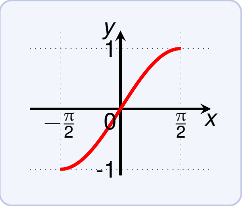


Problem: these functions are not one-to-one!

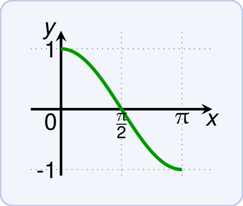
Solution: we restrict their domain

- for sin we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- for cos we restrict the domain to $[0, \pi]$

Inverse Trigonometric Functions



$\sin x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$\cos x$ restricted to $[0, \pi]$

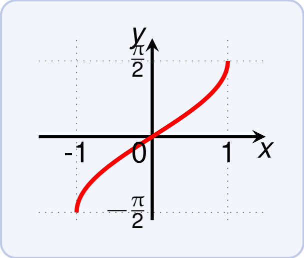
From $\quad f^{-1}(y) = x \iff f(x) = y \quad$ we get:

$$\sin^{-1}(y) = x \quad \iff \quad \sin(x) = y \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

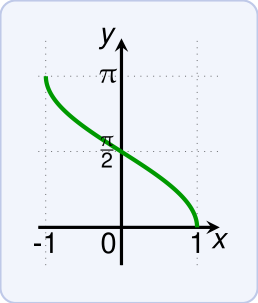
$$\cos^{-1}(y) = x \quad \iff \quad \cos(x) = y \text{ and } 0 \leq x \leq \pi$$

The **inverse sine function** $\sin^{-1}$ is also denoted by arcsin.
The **inverse cosine function** $\sin^{-1}$ is denoted by arccos.

Inverse Trigonometric



arcsin x



arccos x

The domain of arcsin and arccos is $[-1, 1]$.
The range of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and of arccos is $[0, \pi]$.

Inverse Trigonometric: Cancellation Equations

The cancellation equations are:

$$\arcsin(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\arcsin x) = x \quad \text{for } -1 \leq x \leq 1$$

$$\arccos(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$
$$\cos(\arccos x) = x \quad \text{for } -1 \leq x \leq 1$$

Inverse Trigonometric: Examples

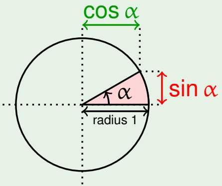
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

$$\sin^{-1}(y) = x \quad \Longleftrightarrow \quad \sin(x) = y \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

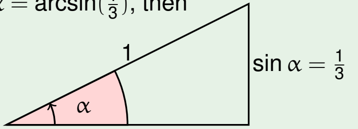
$$\cos^{-1}(y) = x \quad \Longleftrightarrow \quad \cos(x) = y \text{ and } 0 \leq x \leq \pi$$

Evaluate the following:

- $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$
- $\tan(\arcsin(\frac{1}{3})) = \frac{\sin(\arcsin(\frac{1}{3}))}{\cos(\arcsin(\frac{1}{3}))} = \frac{\frac{1}{3}}{\frac{2}{3}\sqrt{2}} = \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

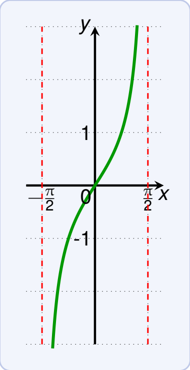


Let $\alpha = \arcsin(\frac{1}{3})$, then

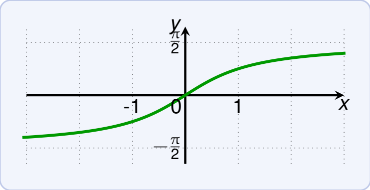


$\cos\alpha = \sqrt{1 - (\frac{1}{3})^2} = \sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$

Trigonometric Functions: Inverse Tangent



$\tan x$ restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$

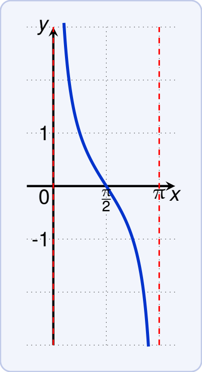


$\tan^{-1} x$ or $\arctan x$

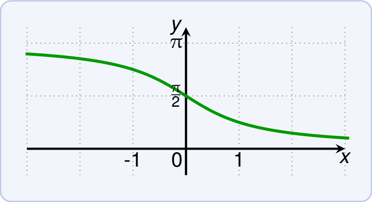
$$\tan^{-1} y = x \iff \tan x = y \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

The function arctan has domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Trigonometric Functions: Inverse Cotangent



$\cot x$ restricted to $(0, \pi)$



$\cot^{-1} x$

$$\cot^{-1} y = x \iff \cot x = y \text{ and } 0 < x < \pi$$

The function $\cot^{-1}$ has domain $(-\infty, \infty)$ and range $(0, \pi)$.

Exercises

Classify the following functions as one of the types that we have discussed:

1. $f(x) = 5^x$ is an exponential function
2. $g(x) = x^5$ is a power function, a polynomial of degree 5, a rational function and an algebraic function.
3. $h(x) = \frac{1+x}{1-\sqrt{x}}$ is an algebraic function.
4. $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4, a rational function and an algebraic function.
5. $v(x) = x^{-3}$ is a power function, a rational function and an algebraic function.
6. $p(x) = x^{-\frac{1}{3}}$ is a power function, and an algebraic function.
7. $z(x) = \frac{1+x}{3+x^2}$ is a rational function, and algebraic function.

Exercises

Assume that a ball is dropped, and we have the following measurements:

- height at time 0s is 490m
- height at time 2s is 472m
- height at time 4s is 414m

Find a quadratic function for the height of the ball after time t. When does the ball hit the ground?

We look for a function of the form:

$$h(t) = at^2 + bt + c$$

We know

$$\begin{aligned} h(0) &= c = 490 \\ h(2) &= 2^2a + 2b + 490 = 472 \\ h(4) &= 4^2a + 4b + 490 = 414 \end{aligned}$$

Exercises

We know $c = 490$ and

$$(1) \quad h(2) = 2^2 a + 2b + 490 = 472$$

$$(2) \quad h(4) = 4^2 a + 4b + 490 = 414$$

We simplify

$$(1) \quad 4a + 2b + 18 = 0$$

$$(2) \quad 16a + 4b + 76 = 0$$

We solve by taking $(2) - 2 \cdot (1)$:

$$h(2) = 8a + 40 = 0 \quad \Longrightarrow \; 8a = -40 \;\Longrightarrow\; a = -5$$

We get b by plugging $a = -5$ in (1):

$$4 \cdot (-5) + 2b + 18 = 0 \quad \Longrightarrow \; 2b = 2 \;\Longrightarrow\; b = 1$$

Thus $h(t) = -5t^2 + t + 490$.

Exercises

Formula for the height:

$$h(t) = -5t^2 + t + 490$$

When does the ball hit the ground? When the height is 0:

$$-5t^2 + t + 490 = 0 \quad \Longrightarrow \; t^2 - \frac{t}{5} - 98 = 0$$

Solving the quadratic formula:

$$t = \frac{1}{10} \pm \sqrt{(\frac{1}{10})^2 + 98} = \frac{1}{10} \pm \sqrt{\frac{1}{100} + \frac{9800}{100}} = \frac{1}{10} \pm \frac{\sqrt{9801}}{10}$$

We know $100^2 = 10000$ and $(100 - n)^2 = 10000 - 200n + n^2$.
Thus $\sqrt{9801} = 99$.

$$t = \frac{1}{10} \pm \frac{99}{10} \quad \Longrightarrow \quad t = 10 \;\text{ or }\; t = -\frac{98}{10}$$

Thus the ball hits the ground after 10 seconds.