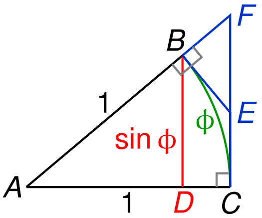
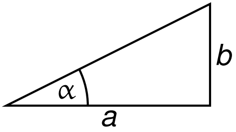
Derivatives of Trigonometric Functions

We investigate $\lim_{\phi\to 0}\frac{\sin\phi}{\phi}$ (for simplicity assume $0 < \phi < \pi/2$)





$$\tan\alpha = \frac{b}{a}$$
$$b = a\cdot\tan\alpha$$

$$\sin\phi = |BD| < \phi \quad\Longrightarrow\quad \frac{\sin\phi}{\phi} < 1$$

$$\phi < |CE| + |EB| \;\;\&\;\; |EB| < |EF| \quad\Longrightarrow\quad \phi < |CE| + |EF| = |CF|$$

$$\phi < |CF| = 1\cdot\tan\phi = \frac{\sin\phi}{\cos\phi} \quad\Longrightarrow\quad \cos\phi < \frac{\sin\phi}{\phi} < 1$$

We use the Squeeze Theorem:

$$\lim_{\phi\to 0}\cos\phi = 1 = \lim_{\phi\to 0} 1 \quad\Longrightarrow\quad \lim_{\phi\to 0}\frac{\sin\phi}{\phi} = 1$$

Derivatives of Trigonometric Functions

We have the following identities for sin and cos:

$$\begin{aligned}\sin(x+y) &= \sin x \cdot \cos y + \cos x \cdot \sin y \\ \cos(x+y) &= \cos x \cdot \cos y - \sin x \cdot \sin y\end{aligned}$$

Derivatives of Trigonometric Functions

We will prove that

$$\frac{d}{dx}\sin(x) = \cos(x)$$

We have

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\cos h-\sin(x)}{h}+\frac{\cos x\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h}\right] \\
&= \sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1}
\end{aligned}$$

Derivatives of Trigonometric Functions

We will prove that

$$\frac{d}{dx}\sin(x) \;=\; \cos(x)$$

We have

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \\
&= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} \right) + \cos x \\
&= \sin x \cdot \lim_{h\to 0} \left(\frac{(\cos h)^2 - 1}{h(\cos h + 1)} \right) + \cos x \\
&= \sin x \cdot \lim_{h\to 0} \left(\frac{-(\sin h)^2}{h(\cos h + 1)} \right) + \cos x \\
&= \sin x \cdot \left(\lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{-\sin h}{\cos h + 1} \right) + \cos x \\
&= \sin x \cdot 1 \cdot 0 + \cos x = \cos x
\end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

Differentiate

$$f(x) = x^2 \sin x$$

We have

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx}\sin x + \sin x \frac{d}{dx}x^2 \quad \text{product rule} \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d}{dx}\cos x = -\sin x$$

Differentiate $\tan x$:

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}\cos x}{(\cos x)^2} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

Differentiate the **secant** $\sec x = \frac{1}{\cos x}$:

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\ &= \frac{\cos x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\cos x}{(\cos x)^2} \\ &= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x\end{aligned}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d}{dx}\cos x = -\sin x$$

Differentiate the **cosecant** $\csc x = \frac{1}{\sin x}$:

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{\sin x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\sin x}{(\sin x)^2}\\ &= \frac{-\cos x}{\sin^2 x} = -\csc x \cdot \cot x\end{aligned}$$

Derivatives of Trigonometric Functions

Summary: derivatives of trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x \qquad \frac{d}{dx}\cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = \sin(x^2)$.

We have $f = g \circ h$ where $\quad g(x) = \sin x \quad$ and $\quad h(x) = x^2 \;$:

$$\begin{aligned} g'(x) &= \cos x \\ h'(x) &= 2x \\ f'(x) &= g'(h(x)) \cdot h'(x) = \cos(x^2) \cdot 2x = 2x\cos(x^2) \end{aligned}$$

Differentiate $g(x) = \sin^2 x = (\sin x)^2$.

We have $f = g \circ h$ where $\quad g(x) = x^2 \quad$ and $\quad h(x) = \sin x \;$:

$$\begin{aligned} g'(x) &= 2x \\ h'(x) &= \cos x \\ f'(x) &= g'(h(x)) \cdot h'(x) = 2\sin x \cdot \cos x \end{aligned}$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = e^{\sin x}$.

We have $f = g \circ h$ where $g(x) = e^x$ and $h(x) = \sin x$:

$$
\begin{aligned}
g'(x) &= e^x \\
h'(x) &= \cos x \\
f'(x) &= g'(h(x)) \cdot h'(x) = e^{\sin x} \cdot \cos x
\end{aligned}
$$

Derivatives of Trigonometric Functions

Differentiate $f(x) = \sin(\cos(\tan x))$.

$$\begin{aligned} f'(x) &= \cos(\ \cos(\tan x)\ ) \cdot \frac{d}{dx}\cos(\tan x) \\ &= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot \frac{d}{dx}\tan x \\ &= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \frac{1}{\cos^2 x} \end{aligned}$$

Note that we have applied the chain rule twice!