Exponential Growth and Decay

Often quantities grow or decay proportional to their size:

- growth of a population (animals, bacteria,...)
- decay of radioactive material
- growth of savings on your bank account (interest rates)

Assume that

- $y(t)$ be a quantity depending on time t
- rate of change of $y(t)$ is proportional to $y(t)$

Then

$$y' = ky \qquad \text{or equivalently} \qquad \frac{d}{dt}y = ky$$

where k is a constant. This equation is called:

- **law of natural growth** if $k > 0$
- **law of natural decay** if $k < 0$

Exponential Growth and Decay

Assume that $y(t)$ be a function, and k a constant such that

$$y' = ky$$

We have seen functions with this behavior:

$$y(t) = Ce^{kt} \qquad\qquad y'(t) = k(Ce^{kt}) = ky(t)$$

Note that

$$y(0) = Ce^0 = C$$

The only solutions of the differential equation

$$y' = ky$$

are the exponential functions

$$y(t) = Ce^{kt}$$

where C is any real number.

Exponential Population Growth

Let y be the size of a population.

Instead of saying 'the growth rate is proportional to the size'

$$y' = ky$$

we can equivalently say that the **relative growth rate**

$$\frac{y'}{y} = k \qquad \text{or equivalently} \qquad \frac{1}{y}\frac{dy}{dt} = k$$

is constant.

Then the solution is of the form

$$y = Ce^{kt}$$

Exponential Population Growth

The world population was

- 2560 million in 1950, and
- 3040 million in 1960.

Assume a constant growth rate. Find a formula $P(t)$ with

- $P(t)$ in millions of people and
- t in years since 1950.

We have

$$P(t) = P(0)e^{kt}$$
$$P(0) = 2560$$
$$P(10) = 2560e^{10k} = 3040$$
$$e^{10k} = \frac{3040}{2560} \quad \Longrightarrow \quad k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017$$

The world population growths with a rate of 1.7% per year.

Exponential Radioactive Decay

Let $m(t)$ be the mass of a radioactive substance after time t.

Then the **relative decay rate** rate

$$-\frac{m'}{m} = k \qquad \text{or equivalently} \qquad -\frac{1}{m}\frac{dm}{dt} = k$$

is constant.

Then the solution is of the form

$$m = Ce^{-kt}$$

Physicists typically express the decay in terms of half-life.

The **half-life** is the time until only half of the quantity is left.

Exponential Radioactive Decay

The half-life of radium-226 is 1590 years.

- We consider a sample of 100mg.

Find a formula for the mass that remains after t years.

We have:

$$m(t) = m(0) \cdot e^{-kt}$$

$$m(0) = 100$$

$$m(1590) = \frac{1}{2} \cdot 100 = 50 = 100 \cdot e^{-k \cdot 1590}$$

$$e^{-k \cdot 1590} = \frac{1}{2} \quad \Longrightarrow \quad -k \cdot 1590 = \ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$$

$$k = \frac{\ln 2}{1590}$$

Hence $m(t) = 100 e^{-\frac{\ln 2}{1590} t} = 100 \left(\frac{1}{2}\right)^{\frac{t}{1590}}$ is the mass after t years.

Newtons Law of Cooling/Warming

Newtons Law of Cooling

The rate of cooling of an object is proportional to the temperature difference of the object and surrounding temperature.

Let

- $T(t)$ be the temperature after time t, and
- T_s the temperature of the surroundings.

Then the law can be written as differential equation:

$$T'(t) = k(T(t) - T_s)$$

where k is constant.

This is not yet the form that we need. Let

$$y(t) = T(t) - T_s \quad \text{then} \quad y'(t) = T'(t) \quad \text{thus} \quad y'(t) = ky(t)$$

Thus the solution for y is an exponential function Ce^{kt}.

Newtons Law of Cooling/Warming

$$T'(t) = k(T(t) - T_s)$$

A bottle of water is placed in the refrigerator:

- bottle has temperature 60°F,
- refrigerator has temperature 20°F

After 2 minutes the bottle has cooled down to 30°F.

- Find a formula for the temperature.

$$T'(t) = k(T(t) - T_s) = k(T(t) - 20)$$

We let $y(t) = T(t) - 20$, then

$$y(0) = T(0) - 20 = 60 - 20 = 40$$
$$y(t) = y(0)e^{kt} = 40e^{kt}$$
$$y(2) = 40e^{k2} = T(2) - 20 = 10 \implies k = \frac{\ln \frac{10}{40}}{2} = \ln \frac{1}{2}$$

Thus $T(t) \;=\; y(t) + 20 \;=\; 40e^{t \cdot \ln \frac{1}{2}} + 20$

Continuously Compounded Interest

Assume 1000$ are invested with 6% interest compounded annually. Then

- after 1 year we have $1000\$ \cdot 1.06 = 1060\$$
- after 2 year we have $1000\$ \cdot 1.06^2 = 1123.6\$$
- after t year we have $1000\$ \cdot 1.06^t$

If A_0 is invested with interest rate r, compounded annually, then after t years the amount is

$$A_0 \cdot (1 + r)^t$$

Usually, interest is compounded more frequently.

If the interest is compounded n times per year, then after t years the value is

$$A_0 \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

Continuously Compounded Interest

If the interest is compounded n times per year, then after t years the value is

$$A_0 \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

For instance, 1000\$ with 6% interest after 3 years:

- $1000\$ \cdot (1 + 0.06)^3 = 1191.02\$$ annual compounding
- $1000\$ \cdot (1 + 0.03)^6 = 1194.05\$$ semiannual compounding
- $1000\$ \cdot (1 + 0.015)^{12} = 1195.62\$$ quarterly compounding
- $1000\$ \cdot (1 + 0.005)^{36} = 1196.68\$$ monthly compounding
- $1000\$ \cdot (1 + 0.06/356)^{356 \cdot 3} = 1197.20\$$ daily compounding

If we let $n \to \infty$, we get **continuous compounding**:

$$A(t) = \lim_{n\to\infty} A_0 \cdot \left(1 + \frac{r}{n}\right)^{nt} = A_0 \cdot \left(\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt} = A_0 \cdot e^{rt}$$

- $1000\$ \cdot e^{0.06 \cdot 3} = 1197.22\$$ continuous compounding