Maximum and Minimum Values

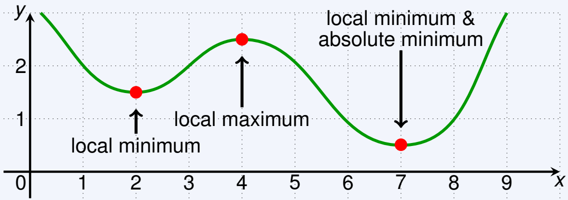
An important application of derivatives are

optimization problems,

that is, finding the best way of doing something.

These problems can often be reduces to finding the minimum or maximum of a function.

Maximum and Minimum Values



Let c be in the domain D of f. Then $f(c)$ is the

- **absolute maximum** value of f if $f(c) \geq f(x)$ for all x in D
- **absolute minimum** value of f if $f(c) \leq f(x)$ for all x in D

Often called **global maximum** or **global minimum.**
Minima and maxima are called **extreme values** of f.

The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c

Maximum and Minimum Values

Where does

$$f(x) = x^2$$

have local / global minima or maxima?

The value $f(0) = 0$ is absolute and local minimum since:

$$f(0) = 0 \leq x^2 = f(x) \quad \text{for all } x$$

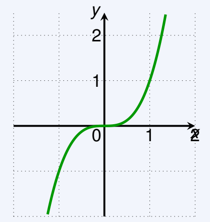
The function has no local or global maxima.

Where does

$$f(x) = x^3$$

have (local or global) minima or maxima?

The function has no local or global extrema.

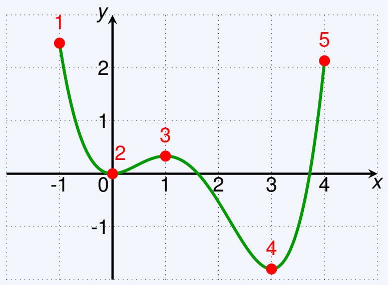


Maximum and Minimum Values

The graph of

$$f(x) = \frac{3x^4 - 16x^3 + 18x^2}{15}$$

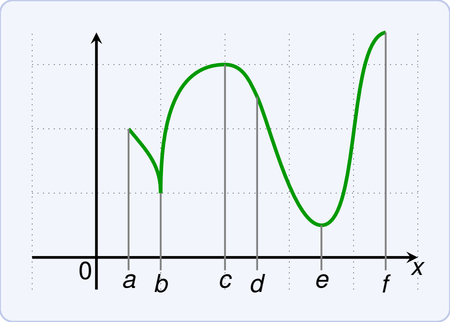
for $-1 \leq x \leq 4$ is shown in this diagram:



Which of the points are a local / global maxima or minima?

1. global (absolute) maximum;
   not a local maximum since f is not defined near -1
2. local minimum
3. local maximum
4. global (absolute) and local minimum
5. nothing

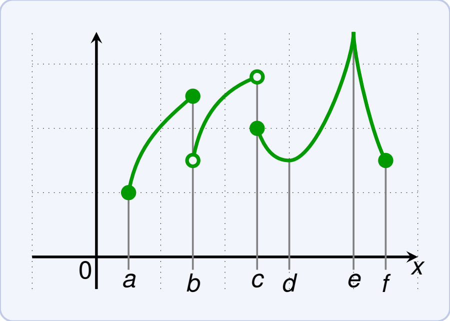
Maximum and Minimum Values



Which of the points are global/local maxima/minima?

- *a* nothing
- *b* local minimum
- *c* local maximum
- *d* nothing
- *e* local and global (absolute) minimum
- *f* global (absolute) maximum, but not a local maximum

Maximum and Minimum Values



Which of the points are global/local maxima/minima?

- *a* global (absolute) minimum, but not a local minimum
- *b* local maximum
- *c* nothing
- *d* local minimum
- *e* local and global (absolute) maximum
- *f* nothing

Maximum and Minimum Values

Let f be a function, and $[a, b]$ a closed interval. Then $f(c)$ is an

- **absolute maximum** on $[a, b]$ if $f(c) \geq f(x)$ for all x in $[a, b]$
- **absolute minimum** on $[a, b]$ if $f(c) \leq f(x)$ for all x in $[a, b]$

Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then

- f has an absolute maximum $f(c)$ for some c in $[a, b]$,
- f has an absolute minimum $f(d)$ for some d in $[a, b]$.

Continuous on $[1, 7]$.

Absolute minimum:
$f(4) = 1$

Absolute maximum:
$f(2) = 3$, and
$f(6) = 3$

Maximum and Minimum Values

Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then

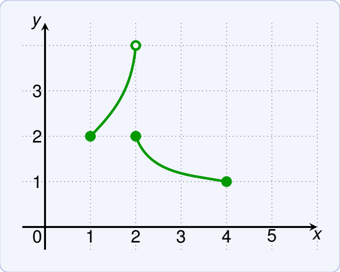
- f has an absolute maximum $f(c)$ for some c in $[a, b]$,
- f has an absolute minimum $f(d)$ for some d in $[a, b]$.

Continuous on $[1, 6]$.

Absolute minimum:
$f(6) = 1$

Absolute maximum:
$f(3) = 3$

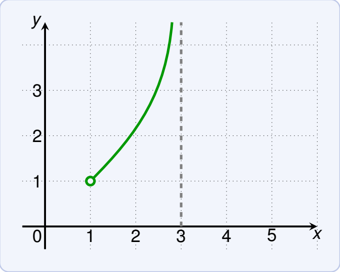
Maximum and Minimum Values



Absolute minimum:
$f(4) = 1$

Absolute maximum:
none

Not continuous on $[1, 4]$!



Absolute minimum:
none

Absolute maximum:
none

Continuous on $(1, 3)$, but this is not a closed interval!

The function needs to be **continuous** on a **closed** interval $[a, b]$.

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

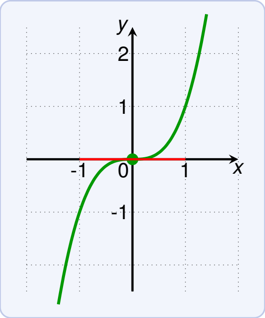
At every **local** maximum or minimum, the tangent is horizontal.
(if the derivative exists)

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

The reverse statement is not true! Having $f'(c) = 0$ does not guarantee that $f(c)$ is a minimum or maximum.



For example:

$$f(x) = x^3$$

Then $f'(0) = 0$.

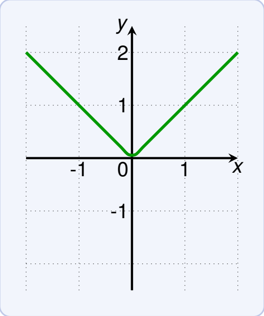
But there is no minimum or maximum.

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

A local minimum/maximum does not guarantee that $f'(c)$ exists.



For example:

$$f(x) = |x|$$

Then $f(0) = 0$ is a local minimum.
But $f'(0)$ does not exist.

Care needed for applying the theorem (check both conditions)!

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

The theorem suggests where local extra can occur:

- where $f'(c) = 0$, or
- where $f'(c)$ does not exist.

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$, or $f'(c)$ does not exist.

What are the critical numbers of $f(x) = x^{3/5}(5 - x)$?

$$f(x) = x^{3/5}(5 - x) = 5x^{3/5} - x^{8/5}$$

$$f'(x) = \frac{3}{x^{2/5}} - \frac{8}{5}x^{3/5} = \frac{15}{5x^{2/5}} - \frac{8x}{5x^{2/5}} = \frac{15 - 8x}{5x^{2/5}}$$

The critical numbers are $\frac{15}{8}$ ($f(c) = 0$) and 0 ($f(c)$ does not exist)

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

What are the critical numbers of the function

$$f(x) = \sqrt{x} + |x - 2| \qquad ?$$

Due to $|x - 2|$, the derivative is not defined at x = 2 .

For $x < 2$ we have $|x - 2| = -(x - 2)$, thus:

$$f(x) = \sqrt{x} - (x - 2) \qquad\qquad f'(x) = \frac{1}{2\sqrt{x}} - 1$$

Thus $f'(x) = 0 \iff x = 1/4$, and $f'(x)$ undefined for $x = 0$.

For $x > 2$ we have $|x - 2| = x - 2$, thus:

$$f(x) = \sqrt{x} + (x - 2) \qquad\qquad f'(x) = \frac{1}{2\sqrt{x}} + 1 \geq 1$$

Thus the critical numbers are 0, $1/4$ and 2.

Maximum and Minimum Values

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

We can now rephrase the the theorem as follows:

If f has a local extremum at c, then c is a critical number of f.

We can use this to look for global extrema on intervals:

Closed Interval Method

To find the **absolute** maximum and minimum values of a continuous function f on an closed interval $[a, b]$:

1. Find the values of f at critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest value of (1) and (2) is the absolute maximum, the lowest the absolute minimum.

Maximum and Minimum Values

Find the absolute absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$

Since f is cont. on $[-\frac{1}{2}, 4]$ we can use Closed Interval Method.

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

We have $f'(x) = 0$ if $x = 0$ or $x = 2$. Both in $[-\frac{1}{2}, 4]$!
No other critical values since $f'(x)$ exists for all x.

The values of f at the critical numbers are:

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the end points of the interval are:

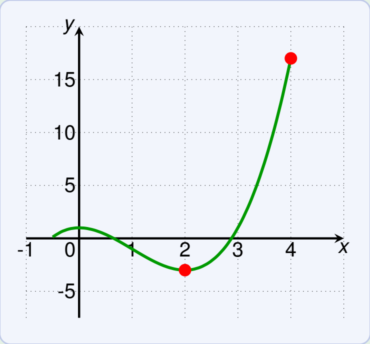
$$f(-\frac{1}{2}) = -\frac{1}{8} - 3\frac{1}{4} + 1 = \frac{1}{8} \qquad f(4) = 4 \cdot 16 - 3 \cdot 16 + 1 = 17$$

Absolute minimum is $f(2) = -3$, absolute maximum $f(4) = 17$.

Maximum and Minimum Values

Find the absolute absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$



Absolute minimum is $f(2) = -3$, absolute maximum $f(4) = 17$.

Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad 0 \le t \le 5$$

Find the absolute minimum and maximum acceleration.

The acceleration is:

$$a(t) = v'(t) = 3(t-1)^2 - 8t + 9 = 3t^2 - 14t + 12$$

Since a is cont. on $[0,5]$ we can use Closed Interval Method.

$$a'(t) = 6t - 14 \qquad a'(t) = 0 \iff t = \frac{7}{3}$$

The only critical number is $\frac{7}{3}$. Note that $\frac{7}{3}$ is in $[0,5]$.
No other critical numbers since $a'(t)$ is defined everywhere.

Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad 0 \le t \le 5$$

Find the absolute minimum and maximum acceleration.

The acceleration is:

$$a(t) = v'(t) = 3t^2 - 14t + 12$$

$$a'(t) = 6t - 14 \qquad\qquad a'(t) = 0 \iff t = \frac{7}{3}$$

The values at critical numbers and end points of the interval:

$$a\left(\frac{7}{3}\right) = 3\left(\frac{7}{3}\right)^2 - 14\frac{7}{3} + 12 = \frac{7 \cdot 7}{3} - \frac{14 \cdot 7}{3} + \frac{36}{3} = -\frac{13}{3}$$

$$a(0) = 12$$

$$a(5) = 3 \cdot 5^2 - 14 \cdot 5 + 12 = 15 \cdot 5 - 14 \cdot 5 + 12 = 17$$

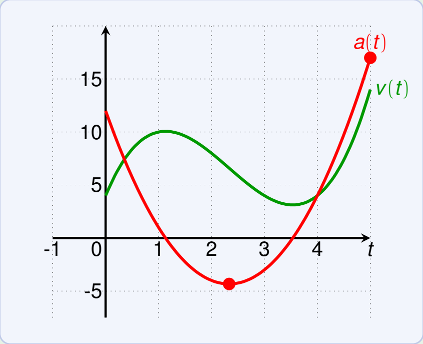
The absolute minimum acceleration is $a\left(\frac{7}{3}\right) = -\frac{13}{3}$.
The absolute maximum acceleration is $a(5) = 17$.

Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad 0 \le t \le 5$$

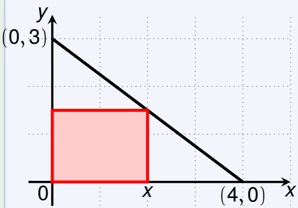
Find the absolute minimum and maximum acceleration.



The absolute minimum acceleration is $a\left(\frac{7}{3}\right) = -\frac{13}{3}$.
The absolute maximum acceleration is $a(5) = 17$.

Exam Task from 2003

Find the area of the largest rectangle that can be inscribed as shown in the triangle.



The line trough $(0,3)$ & $(4,0)$ has the equation: $\ell(x) = -\frac{3}{4}x + 3$

The area A of the rectangle depends on the width x:

$$A(x) = x \cdot \ell(x) = x \cdot (-\frac{3}{4}x + 3) = -\frac{3}{4}x^2 + 3x \quad \text{for } x \text{ in } [0,4]$$

$$A'(x) = -\frac{3}{2}x + 3 \qquad A'(x) = 0 \iff \frac{3}{2}x = 3 \iff x = 2$$

Thus the only critical number is 2. The value of $A(x)$ at 0, 2, 4:

$$A(0) = 0 \qquad A(2) = 3 \qquad A(4) = 0$$

The the area of the largest rectangle is 3.