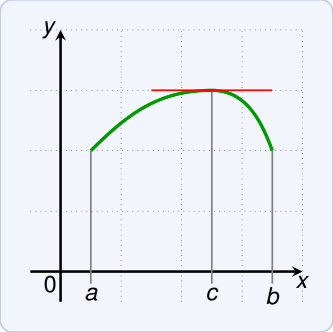
Mean Value Theorem

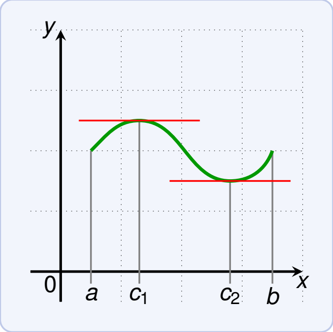
Rolle's Theorem

Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.





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Proof.

- If f is constant, then $f'(c) = 0$ for all c in (a, b).
- If f is not constant, then there is x in (a, b) such that
  $$f(x) > f(a) \qquad \text{or} \qquad f(x) < f(a)$$
  Assume $f(x) > f(a)$. By the Extreme Value Theorem there is a c in $[a, b]$ such that $f(c)$ is the absolute maximum.

  Then c must be in (a, b) and hence is a local maximum. Hence $f'(c) = 0$ by Fermat's Theorem. $\square$

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Let $s(t)$ be the position of an object after time t.
The object is in the same place at time $t = 2s$ and $t = 10s$.

What does Rolle's Theorem tell us about the object?

It tells that there is a time c between $2s$ and $10s$ such that the

$$s'(t) = 0$$

that is, the velocity of the object at time c is 0.

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Then there is a number c in (a, b) such that $f'(c) = 0$.

Show that the function f is one-to-one (never takes the same value twice):

$$f(x) = x^3 + x - 1$$

Assume there would be $x_1 < x_2$ such that $f(x_1) = f(x_2)$.

The function f is continuous and differentiable on $[x_1, x_2]$.

By Rolle's Theorem there exists c in (x_1, x_2) with $f'(c) = 0$.

This is a contradiction since $f'(x) = 3x^2 + 1 \geq 1$ for all x.
There no $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Thus f is one-to-one.

Mean Value Theorem

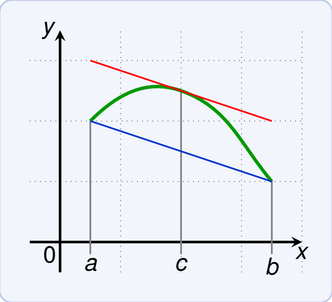
Mean Value Theorem

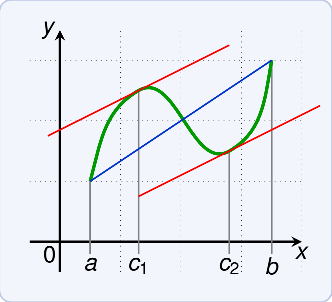
Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ this is the slope from $(a, f(a))$ to $(b, f(b))$



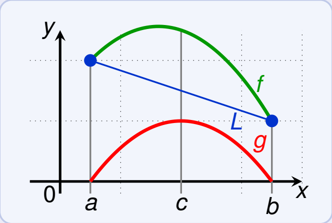


Mean Value Theorem

Proof of the Mean Value Theorem

Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)



Let $L = mx + n$ be the line through $(a, f(a))$ and $(b, f(b))$.

Define $g = f - L$. Then $g(a) = 0$ and $g(b) = 0$.

By Rolle's Theorem there is c in (a, b) such that $g'(c) = 0$.

Since $f = g + L$ we get $f'(c) = g'(c) + m = m = \frac{f(b)-f(a)}{b-a}$.

Mean Value Theorem

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Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Consider the function

$$f(x) = x^3 - x$$

on the interval $[a, b]$ with $a = 0$ and $b = 2$.

This is a polynomial, thus continuous and differentiable on $[0, 2]$.

By the Mean Value Theorem, there is a c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{6}{2} = 3$$

Indeed, we can find such a c, namely: $f'\left(\frac{2}{\sqrt{3}}\right) = 3$.

Mean Value Theorem

Mean Value Theorem

Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
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Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let $s(t)$ be the position of an object after time t.

Then the average velocity between time $t = a$ and $t = b$ is:

$$\frac{s(b) - s(a)}{b - a}$$

What does the Mean Value Theorem tell us?

It states that there is a time c between a and b such that

$$f'(c) = \frac{s(b) - s(a)}{b - a} \quad , \quad \text{that is}$$

the instantaneous velocity at c is equal to the average velocity.

Mean Value Theorem

Mean Value Theorem

Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We can interpret the Mean Value Theorem as follows:

> There is a number c in the interval (a, b) such that the instantaneous rate of change at c is equal to the average rate of change over the interval $[a, b]$.

Mean Value Theorem

Mean Value Theorem

Let f be a function satisfying the all of the following:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x.
How large can $f(2)$ possibly be?

By assumption, f is differentiable, and hence continuous.

By the Mean Value Theorem for the interval $[0, 2]$:

There exists c in $(0, 2)$ such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{f(2) + 3}{2}$.

We have:

$$5 \geq f'(c) = \frac{f(2)+3}{2} \implies 10 \geq f(2) + 3 \implies 7 \geq f(2)$$

Thus the largest possible value for $f(2)$ is 7.

Mean Value Theorem

Mean Value Theorem

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- f is differentiable on (a, b)

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Important consequences of the Mean Value Theorem are:

If $f'(x) = 0$ for all x in (a, b) then f is constant on (a, b).

(Proof like the previous example)

If $f'(x) = g'(x)$ for all x in (a, b) then $f - g$ is constant on (a, b).

(In other words, then $f(x) = g(x) + k$ for a constant k)

Proof.

Let $h = f - g$. Then $h' = f' - g' = 0$ on (a, b). Thus h is constant on (a, b). □