# Derivatives and the Shape of a Graph

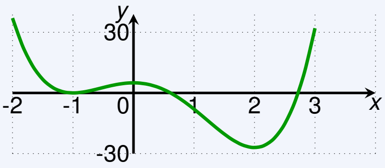
If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.

If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

Where is $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing/decreasing?

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

| Interval | $12x$ | $x-2$ | $x+1$ | $f'(x)$ | |
|---|---|---|---|---|---|
| x < -1 | - | - | - | - | decreasing on $(-\infty, -1)$ |
| -1 < x < 0 | - | - | + | + | increasing on $(-1, 0)$ |
| 0 < x < 2 | + | - | + | - | decreasing on $(0, 2)$ |
| 2 < x | + | + | + | + | increasing on $(2, \infty)$ |



Recall Fermat's Theorem

If $f$ has a local extremum at $c$, then $c$ is a critical number.

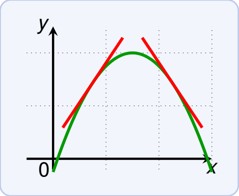
But not ever critical number is an extremum. We need a test!

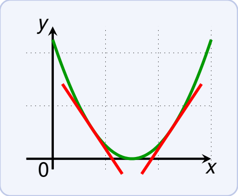
# Derivatives and the Shape of a Graph

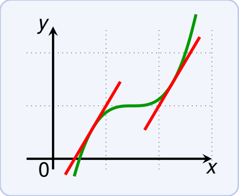
## First Derivative Test

Suppose that $c$ is a critical number of a continuous function $f$.

- If $f'$ changes the sign from positive to negative, then $f$ has a local maximum at $c$.
- If $f'$ changes the sign from negative to positive, then $f$ has a local minimum at $c$.
- If $f'$ does not change sign at $c$, then $f$ has no local extremum at $c$.



# Derivatives and the Shape of a Graph

What are the local extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$?

$$f'(x) = 12x(x-2)(x+1)$$

The critical numbers are: $-1$, 0 and 2.

We have already seen that:

| Interval | $12x$ | $x-2$ | $x+1$ | $f'(x)$ | |
|---|---|---|---|---|---|
| x < -1 | - | - | - | - | decreasing on $(-\infty, -1)$ |
| -1 < x < 0 | - | - | + | + | increasing on $(-1, 0)$ |
| 0 < x < 2 | + | - | + | - | decreasing on $(0, 2)$ |
| 2 < x | + | + | + | + | increasing on $(2, \infty)$ |

We have:

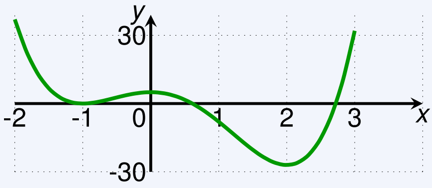
- $f(-1) = 0$ is a local minimum ($f'$ changes from $-$ to $+$)
- $f(0) = 5$ is a local maximum ($f'$ changes from $+$ to $-$)
- $f(2) = -27$ is a local minimum ($f'$ changes from $-$ to $+$)

# Derivatives and the Shape of a Graph

What are the local extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$?

$$f'(x) = 12x(x-2)(x+1)$$

The critical numbers are: $-1$, $0$ and $2$.




We have:

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- $f(0) = 5$ is a local maximum (f′ changes from $+$ to $-$)
- $f(2) = -27$ is a local minimum (f′ changes from $-$ to $+$)

# Derivatives and the Shape of a Graph

What are the local extrema of

$$f(x) = x + 2\sin x \qquad 0 \leq x \leq 2\pi \quad ?$$

We have

$$f'(x) = 1 + 2\cos x$$

$$f'(x) = 0 \iff \cos x = -\frac{1}{2} \iff x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

As $f'$ is defined everywhere these are the only critical numbers.

| Interval | $f'(x)$ | |
|---|---|---|
| $0 < x < \frac{2\pi}{3}$ | + | increasing on $(0, \frac{2\pi}{3})$ |
| $\frac{2\pi}{3} < x < \frac{4\pi}{3}$ | - | decreasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$ |
| $\frac{4\pi}{3} < x < 2\pi$ | + | increasing on $(\frac{4\pi}{3}, 2\pi)$ |

As a consequence:

- $f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum ($f'$ from $+$ to $-$)
- $f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum ($f'$ from $-$ to $+$)

# Derivatives and the Shape of a Graph

What are the local extrema of

$$f(x) = x + 2\sin x \qquad\qquad 0 \le x \le 2\pi \quad ?$$

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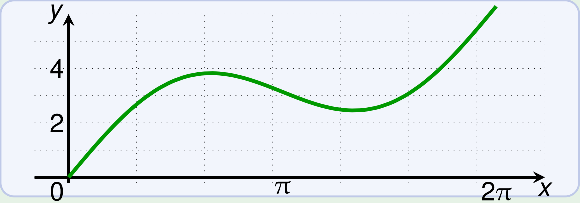


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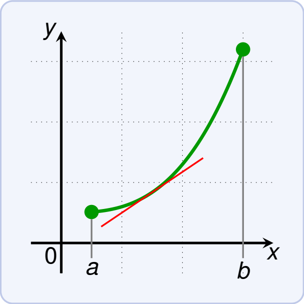
- $f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum ($f'$ from $-$ to $+$)

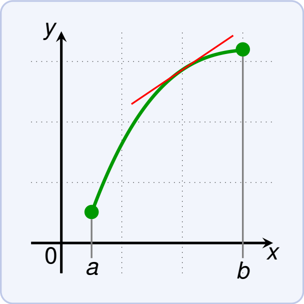
# Derivatives and the Shape of a Graph

Let $I$ be an interval. If the graph of $f$ is called

- **concave up** on $I$ if it it lies above all its tangents on $I$
- **concave down** on $I$ if it it lies below all its tangents on $I$




concave up

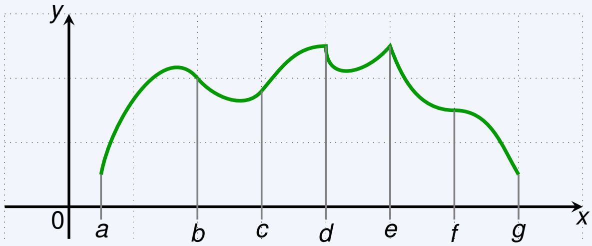



concave down

Imagine the graph as a street & a car driving from left to right:

- then concave upward = turning left (increasing slope)
- then concave downward = turning right (decreasing slope)

# Derivatives and the Shape of a Graph




On which interval is the curve concave up / concave down?

- on (a,b) concave downward
- on (b,c) concave upward
- on (c,d) concave downward
- on (d,e) concave upward
- on (e,f) concave upward
- on (f,g) concave downward

# Derivatives and the Shape of a Graph

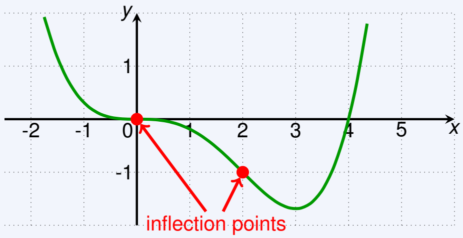
## Concavity Test

If $f''(x) > 0$ for all $x$ in $I$, then $f$ is concave upward on $I$.

If $f''(x) < 0$ for all $x$ in $I$, then $f$ is concave downward on $I$.

A point $P$ on a curve $f(x)$ is called **inflection point** if $f$ is continuous at this point and the curve

- changes from concave upward to downward at $P$, or
- changes from concave downward to upward at $P$.



# Derivatives and the Shape of a Graph

Where are inflection points of $f(x) = x^4 - 4x^3$?

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus $f''(x) = 0$ for $x = 0$ and $x = 2$.

| Interval | $f''(x)$ | |
|---|---|---|
| $x < 0$ | + | concave upward on $(-\infty, 0)$ |
| $0 < x < 2$ | - | concave downward on $(0, 2)$ |
| $2 < x$ | + | concave upward on $(2, \infty)$ |

Thus the inflection points are:

- $(0, 0)$ since the curve changes from concave up to down
- $(2, -16)$ since the curve changes from concave down to up

# Derivatives and the Shape of a Graph

## Second Derivative Test

Suppose $f''$ is continuous near $c$.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

Where does $f(x) = x^4 - 4x^3$ have local extrema?

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus $f'(x) = 0$ for $x = 0$ and $x = 3$. Second Derivative Test:

$$f''(0) = 0 \qquad\qquad f''(3) = 36 > 0$$

Thus $f(3) = -27$ is a local minimum as $f'(3) = 0$ and $f''(3) > 0$.

The Second Derivative Test gives no information for $f''(0) = 0$.

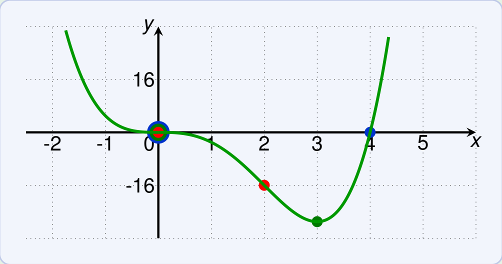
However, the First Derivative Test . . . yields that $f(0) = 0$ is no extremum since $f'(x) < 0$ for $x < 0$ and $0 < x < 3$.

# Derivatives and the Shape of a Graph

## Curve Sketching

$$f(x) = x^4 - 4x^3 = x^3(x-4) \qquad f'(x) = 4x^2(x-3)$$

- $f(x) = 0 \iff x = 0$ or $x = 4$
- local minimum at $(3, -27)$ and $f'(0) = 0$
- inflection points $(0, 0)$ and $(2, -16)$
- decreasing on $(-\infty, 0)$ and $(0, 3)$, increasing on $(3, \infty)$
- concave up on $(-\infty, 0)$, down on $(0, 2)$, up on $(2, \infty)$



# Derivatives and the Shape of a Graph

## Summary: Finding Local Extrema

Find critical numbers $c$: $f'(c) = 0$ or $f'(c)$ does not exist.

First **Derivative Test** ($f$ needs to be continuous at $c$):

- If $f'$ changes from $+$ to $-$ at $c \implies$ local maximum
- If $f'$ changes from $-$ to $+$ at $c \implies$ local minimum
- If $f'$ does not change sign at $c \implies$ no local extremum

The **Second Derivative Test**:

1. $f'(c) = 0$ and $f''(c) > 0 \implies$ local minimum
2. $f'(c) = 0$ and $f''(c) < 0 \implies$ local maximum
3. $f'(c)$ or $f''(c)$ does not exist or $f''(c) = 0$
   $\implies$ use the First Derivative Test