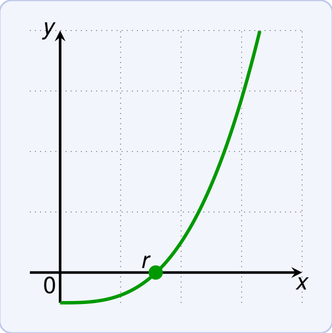
Newton's Method



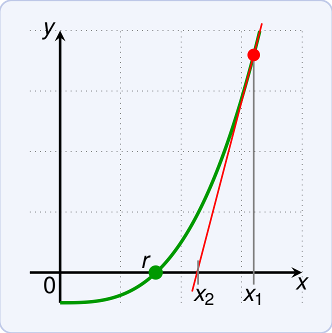
Assume we want to find a root of a complicated function like:

$$f(x) = x^7 - x + \cos x$$

Often it is impossible to solve such equations! E.g. there are no formulas for solutions of polynomials of degree of ≥ 5.

Can we at least find the root approximately?

Newton's Method



Idea of Newton's Method

- Take an approximation x_1 of the root (a rough guess).
- Compute the tangent L_1 at $(x_1, f(x_1))$.
- The tangent L_1 is close to the curve... so x-intercept of L_1 will be close the the x-intercept of the function.

We can repeat this procedure to get improve the approximation.

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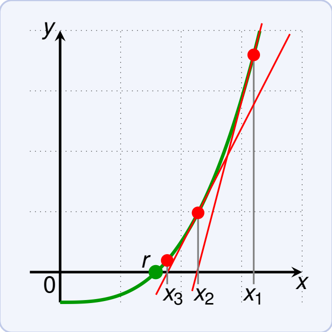
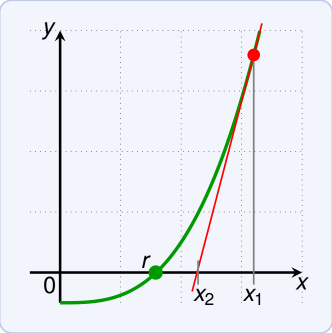


We want to find an approximation of the root r of $f(x)$.

- Take an approximation x_1 of the root (a rough guess).
- Compute the tangent L_1 at $(x_1, f(x_1))$.
- Find the x-intercept x_2 of the tangent L_1.
- Compute the tangent L_2 at $(x_2, f(x_2))$.
- Find the x-intercept x_3 of the tangent L_2.
- …continue until approximation is good enough

Newton's Method



How can we compute x_2? The tangent at $(x_1, f(x_1))$ is

$$y = f(x_1) + f'(x_1)(x - x_1)$$

For the x-intercept x_2 of the tangent, we have:

$$0 = f(x_1) + f'(x_1)(x_2 - x_1) \implies x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We can repeat this process to get x_3, x_4, x_5...

Newton's Method

Newton's Method

Let $f(x)$ be a function, and x_1 and approximation of a root r.

We compute a sequence $x_2, x_3, x_4, \ldots$ of approximations by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The hope is that $x_2, x_3, \ldots$ get closer and closer to the root r.

Let $x_1 = 2$. Find the 3rd approximation to the root of $x^2 - 1$.

$$f'(x) = 2x$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{4} = \frac{5}{4} = 1.25$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{5}{4} - \frac{f(\frac{5}{4})}{f'(\frac{5}{4})} = \frac{5}{4} - \frac{\left(\frac{5}{4}\right)^2 - 1}{\frac{10}{4}} = \frac{41}{40} = 1.025$$

The sequence $x_1, x_2, x_3, \ldots$ gets closer and closer to the root 1.

Newton's Method

Newton's Method

Let $f(x)$ be a function, and x_1 and approximation of a root r.

We compute a sequence $x_2, x_3, x_4, \ldots$ of approximations by

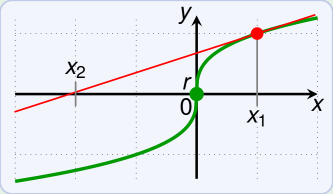
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The hope is that $x_2, x_3, \ldots$ get closer and closer to the root r.
However, this does not always work.

Let $x_1 = 1$. Find the 2nd approximation to the root of $\sqrt[3]{x}$.

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{\left(\frac{1}{3}\right)} = -2$$



Note that $x_2 = -2$ is further away from the root 0 than $x_1 = 1$.

Newton's Method

Newton's Method

Let $f(x)$ be a function, and x_1 and approximation of a root r.

We compute a sequence $x_2, x_3, x_4, \ldots$ of approximations by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The hope is that $x_2, x_3, \ldots$ get closer and closer to the root r.
However, this does not always work.

For more complicated examples see

- Chapter 4.8, Examples 1,2 and 3