Motion in Two Dimensions

- Go over vector and vector algebra
- Displacement and position in 2-D
- Average and instantaneous velocity in 2-D
- Average and instantaneous acceleration in 2-D
- Projectile motion
- Uniform circle motion
- Relative velocity*

Vector and its components

- The components are the legs of the right triangle whose hypotenuse is *A*

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

$$\begin{cases} A_x = A\cos(\theta) \\ A_y = A\sin(\theta) \end{cases}$$

$$\begin{cases} \left|\vec{A}\right| = \sqrt{(A_x)^2 + (A_y)^2} \\ \tan(\theta) = \dfrac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\dfrac{A_y}{A_x}\right) \end{cases}$$

Motion in two dimensions

- Kinematic variables in one dimension
  - Position: $x(t)$ m
  - Velocity: $v(t)$ m/s
  - Acceleration: $a(t)$ m/s^2

- Kinematic variables in three dimensions
  - Position: $\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$ m
  - Velocity: $\vec{v}(t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ m/s
  - Acceleration: $\vec{a}(t) = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ m/s^2

- All are vectors: have direction and magnitudes

Position and Displacement

- In one dimension

$$\Delta x = x_2(t_2) - x_1(t_1)$$

$x_1(t_1) = -4.0$ m, $x_2(t_2) = +2.0$ m

$\Delta x = +2.0\text{ m} + 4.0\text{ m} = +6.0\text{ m}$

- In two dimensions
  - Position: the position of an object is described by its position vector $\vec{r}(t)$ always points to particle from origin.
  - Displacement: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

$$\Delta\vec{r} = (x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})$$
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$
$$= \Delta x\hat{i} + \Delta y\hat{j}$$

Average & Instantaneous Velocity

- Average velocity $\vec{v}_{avg} \equiv \dfrac{\Delta \vec{r}}{\Delta t}$

$$\vec{v}_{avg} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} = v_{avg,x}\hat{i} + v_{avg,y}\hat{j}$$

- Instantaneous velocity

$$\vec{v} \equiv \lim_{t \to 0} \vec{v}_{avg} = \lim_{t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

- $\boldsymbol{v}$ is tangent to the path in x-y graph;

Average & Instantaneous Acceleration

- Average acceleration $$\vec{a}_{avg} \equiv \frac{\Delta\vec{v}}{\Delta t}$$

$$\vec{a}_{avg} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} = a_{avg,x}\hat{i} + a_{avg,y}\hat{j}$$

Figure 4.1
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- Instantaneous acceleration

$$\vec{a} \equiv \lim_{t\to 0} \vec{a}_{avg} = \lim_{t\to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j}$$

- The magnitude of the velocity (the speed) can change
- The direction of the velocity can change, even though the magnitude is constant
- Both the magnitude and the direction can change

Motion in two dimensions

- Motions in three dimensions are independent components
- Constant acceleration equations

$$\vec{v} = \vec{v}_0 + \vec{a}t \qquad \vec{r} - \vec{r} = \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

- Constant acceleration equations hold in each dimension

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t$$

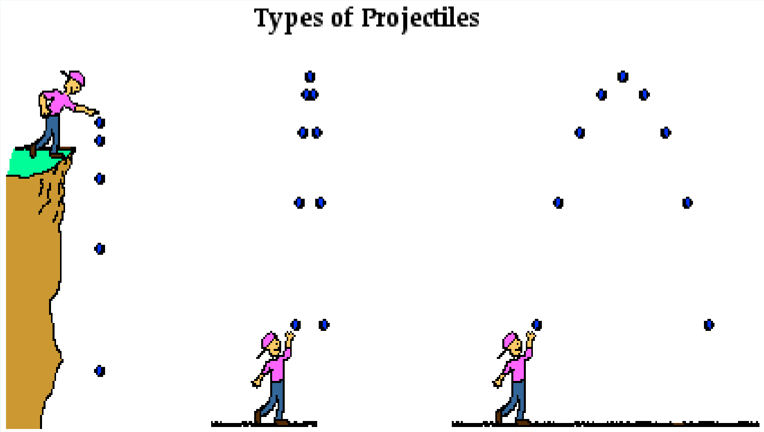
$$x - x_0 = v_{0x} t + \tfrac{1}{2} a_x t^2 \qquad y - y_0 = v_{0y} t + \tfrac{1}{2} a_y t^2$$

$$v_x{}^2 = v_{0x}{}^2 + 2a_x(x - x_0) \qquad v_y{}^2 = v_{0y}{}^2 + 2a_y(y - y_0)$$

  - *t = 0* beginning of the process;
  - $\vec{a} = a_x\hat{i} + a_y\hat{j}$ where a_x and a_y are constant;
  - Initial velocity $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ initial displacement $\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$;

Projectile Motion

- x- horizontal, y- vertical (up +)
- Try to pick $x_0 = 0$, $y_0 = 0$ at $t = 0$
- Horizontal motion + Vertical motion
- Horizontal: $a_x = 0$, constant velocity motion
- Vertical: $a_y = -g = -9.8\ m/s^2$, $v_{0y} = 0$
- Equations:



Horizontal

$$v_x = v_{0x} + a_x t$$

$$x - x_0 = v_{0x} t + \tfrac{1}{2} a_x t^2$$

$$v_x^2 = v_{0x}^2 + 2a_x (x - x_0)$$

Vertical

$$v_y = v_{0y} + a_y t$$

$$y - y_0 = v_{0y} t + \tfrac{1}{2} a_y t^2$$

$$v_y^2 = v_{0y}^2 + 2a_y (y - y_0)$$

$$y_f = y_i + v_{iy} t - \tfrac{1}{2} g t^2$$

Projectile Motion

$$v_x = v_{0x}$$

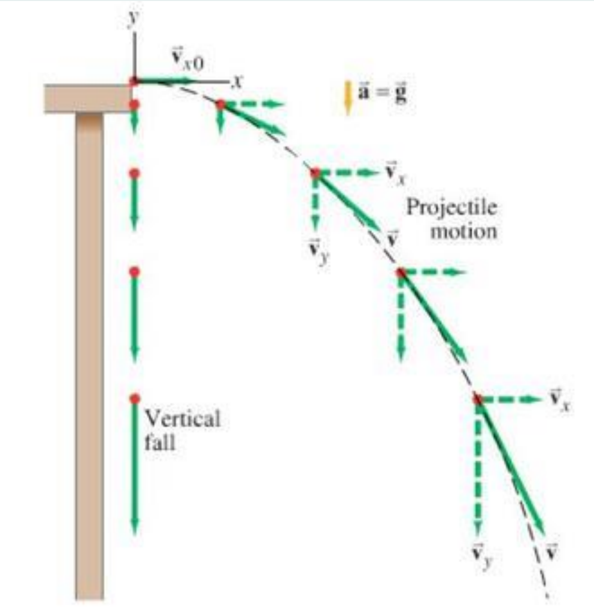
$$x = x_0 + v_{0x}t$$

$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2$$

Horizontal   Vertical

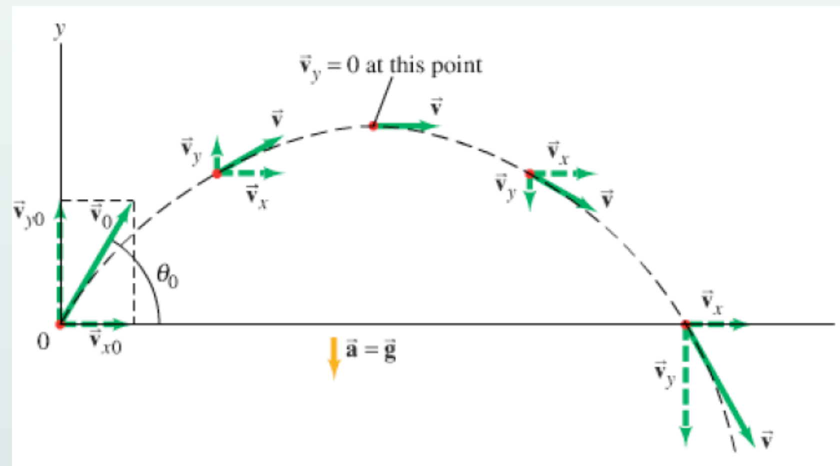
- take $x_0 = 0$, $y_0 = 0$ at $t = 0$
- Horizontal motion + Vertical motion
- Horizontal: $a_x = 0$, constant velocity motion
- Vertical: $a_y = -g = -9.8\ m/s^2$
- x and y are connected by time t
- y(x) is a parabola



Projectile Motion

- Horizontal: $a_x = 0$ and vertical: $a_y = -g$.
- Try to pick $x_0 = 0$, $y_0 = 0$ at $t = 0$.
- Velocity initial conditions:
  - v_0 can have x, y components.
  - v_{0x} is constant usually.
  - v_{0y} changes continuously.
- Equations:

$$v_{0x} = v_0 \sin\theta_0 \qquad v_{0x} = v_0 \cos\theta_0$$



Horizontal

$$v_x = v_{0x}$$

$$x = x_0 + v_{0x} t$$

Vertical

$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y} t - \tfrac{1}{2} g t^2$$

Trajectory of Projectile Motion

- Initial conditions ($t = 0$): $x_0 = 0$, $y_0 = 0$
  $v_{0x} = v_0 \cos\theta_0$ and $v_{0y} = v_0 \sin\theta_0$
- Horizontal motion:

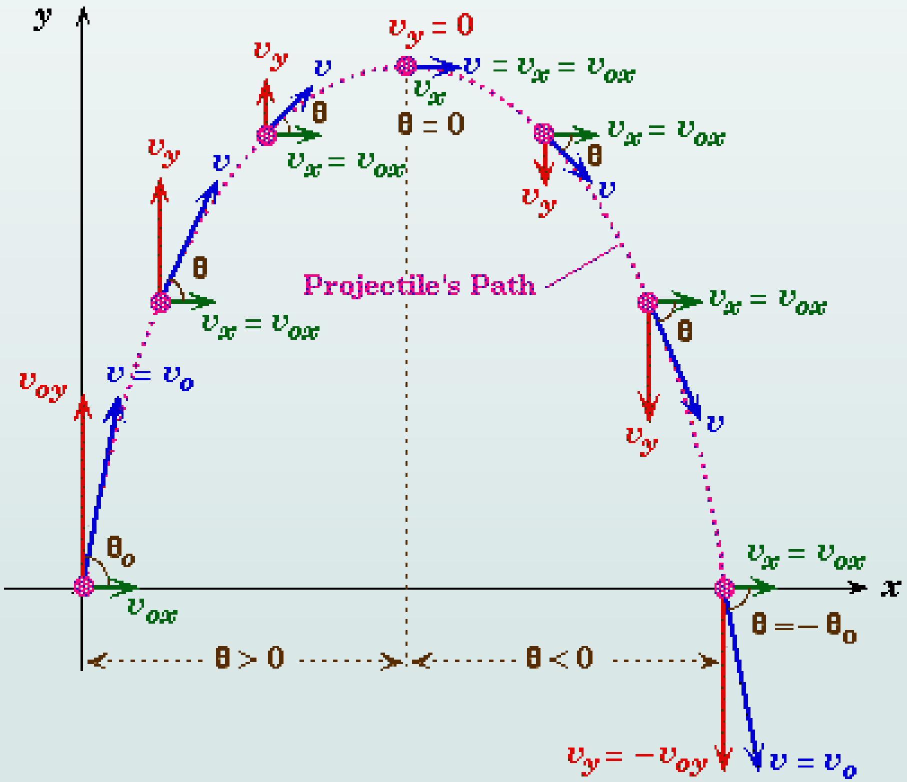
$$x = 0 + v_{0x}t \quad \Rightarrow \quad t = \frac{x}{v_{0x}}$$

- Vertical motion:

$$y = 0 + v_{0y}t - \tfrac{1}{2}gt^2$$

$$y = v_{0y}\left(\frac{x}{v_{0x}}\right) - \frac{g}{2}\left(\frac{x}{v_{0x}}\right)^2$$

$$y = x\tan\theta_0 - \frac{g}{2{v_0}^2\cos^2\theta_0}x^2$$



- Parabola;
  - $\theta_0 = 0$ and $\theta_0 = 90$?

What is R and h ?

- Initial conditions ($t = 0$): $x_0 = 0$, $y_0 = 0$, $v_{0x} = v_0 \cos\theta_0$ and $v_{0x} = v_0 \sin\theta_0$, then

$$x = 0 + v_{0x}t \qquad 0 = 0 + v_{0y}t - \tfrac{1}{2}gt^2$$

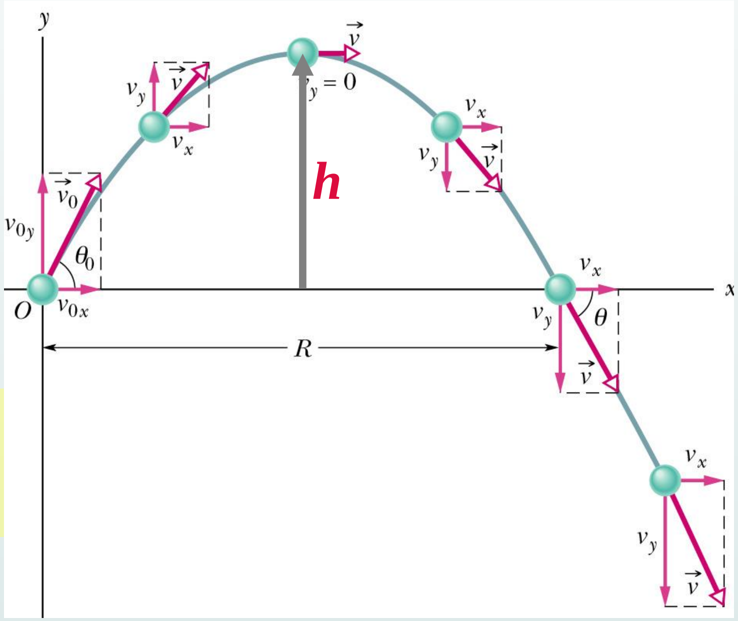
$$t = \frac{2v_{0y}}{g} = \frac{2v_0 \sin\theta_0}{g}$$

$$R = x - x_0 = v_{0x}t = \frac{2v_0 \cos\theta_0 v_0 \sin\theta_0}{g} = \frac{v_0{}^2 \sin 2\theta_0}{g}$$

$$h = y - y_0 = v_{0y}t_h - \tfrac{1}{2}gt_h{}^2 = v_{0y}\frac{t}{2} - \frac{g}{2}\left(\frac{t}{2}\right)^2$$

$$h = \frac{v_0{}^2 \sin^2\theta_0}{2g}$$

$$v_y = v_{0y} - gt = v_{0y} - g\frac{2v_{0y}}{g} = -v_{0y}$$



Horizontal	Vertical
$v_x = v_{0x}$	$v_y = v_{0y} - gt$
$x = x_0 + v_{0x}t$	$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2$

Projectile Motion at Various Initial Angles

- Complementary values of the initial angle result in the same range
  - The heights will be different
- The maximum range occurs at a projection angle of 45°

$$R = \frac{v_0{}^2 \sin 2\phi}{g}$$

Figure 4.11
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Summary

- Position $\vec{r}(t) = x\hat{i} + y\hat{j}$
- Average velocity $\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} = v_{avg,x}\hat{i} + v_{avg,y}\hat{j}$
- Instantaneous velocity $v_x \equiv \frac{dx}{dt}$ $v_y \equiv \frac{dy}{dt}$

$$\vec{v}(t) = \lim_{t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

- Acceleration $a_x \equiv \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$ $a_y \equiv \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$

$$\vec{a}(t) = \lim_{t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j}$$

- $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ are not same direction.