Ankara University
Engineering Faculty
Department of Engineering Physics

PEN207

Circuit Design and Analysis

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Chapter-2

Methods of Circuit Analysis and Circuit Theorems (1/3)

Methods of Analysis and Circuit Theorems
Content

- Direct Implementation of Basic Laws
- Source Representation and Conversion
- Mesh Analysis
- Nodal Analysis
- Mesh and Nodal Analysis in Circuits including Dependent Power Sources
- Y-Δ and Δ-Y Conversions
- Superposition Principle
- Thevenin's and Norton's Theorem

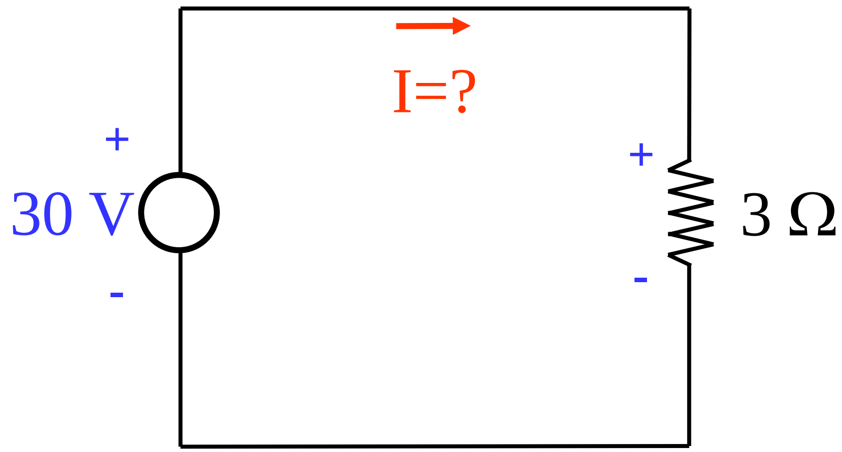
In this chapter,

- Circuits including only resistors will be analyzed,
- Systematic methods for circuit analysis will be developed,
  - Direct Implementation of Basic Laws
  - Mesh Analysis
  - Nodal Analysis
- Source conversion,
- Thevenin’s and Norton’s Theorems

will be learned.

Motivation

What is the current in the circuit below?

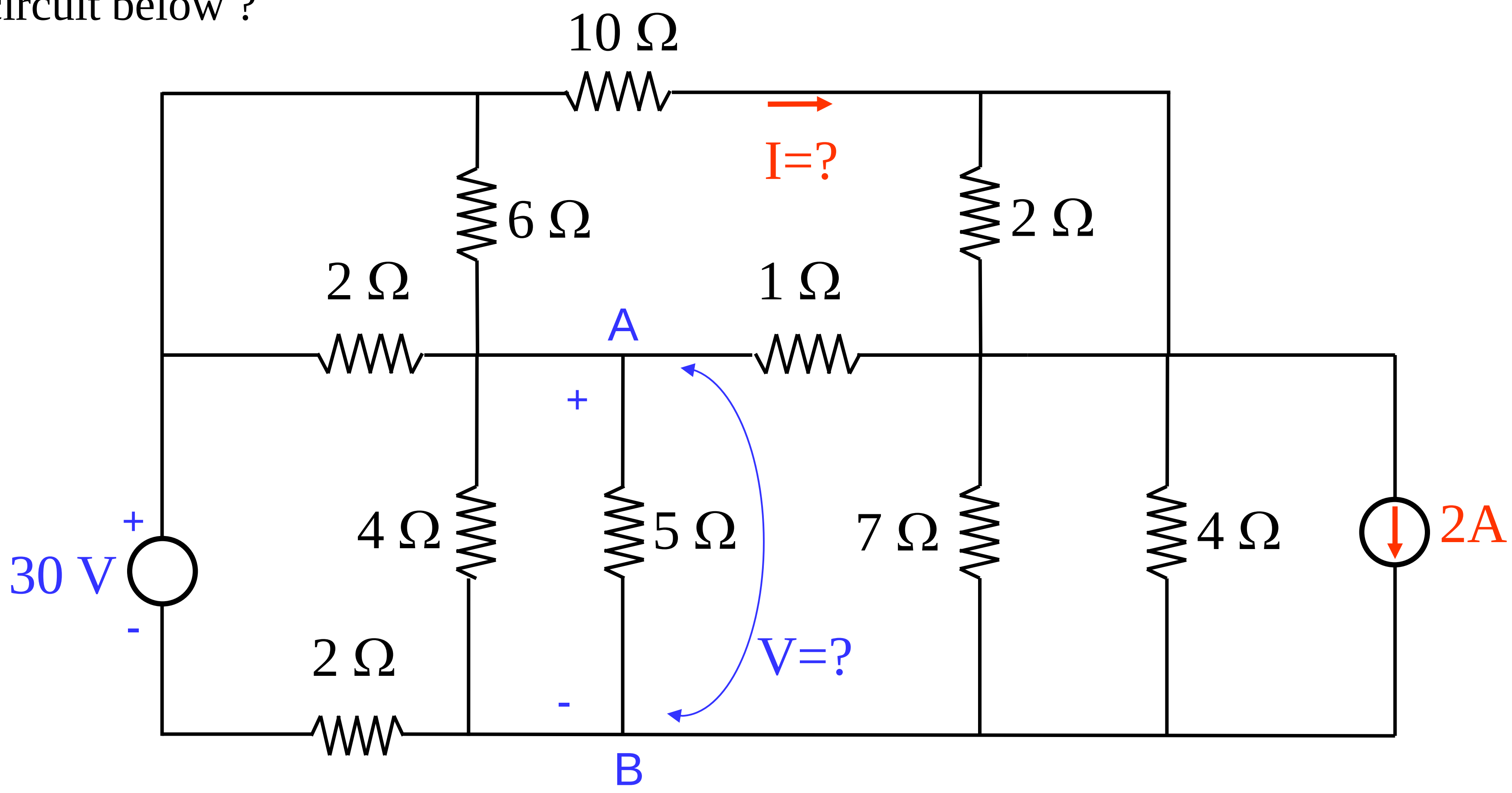


$$V = IR$$

$$I = \frac{V}{R} = \frac{30V}{3\Omega} = 10A$$

Motivation

What is the voltage between points A and B and the current passing over 10Ω in the circuit below ?



Solving this circuit is not as easy as it seems?

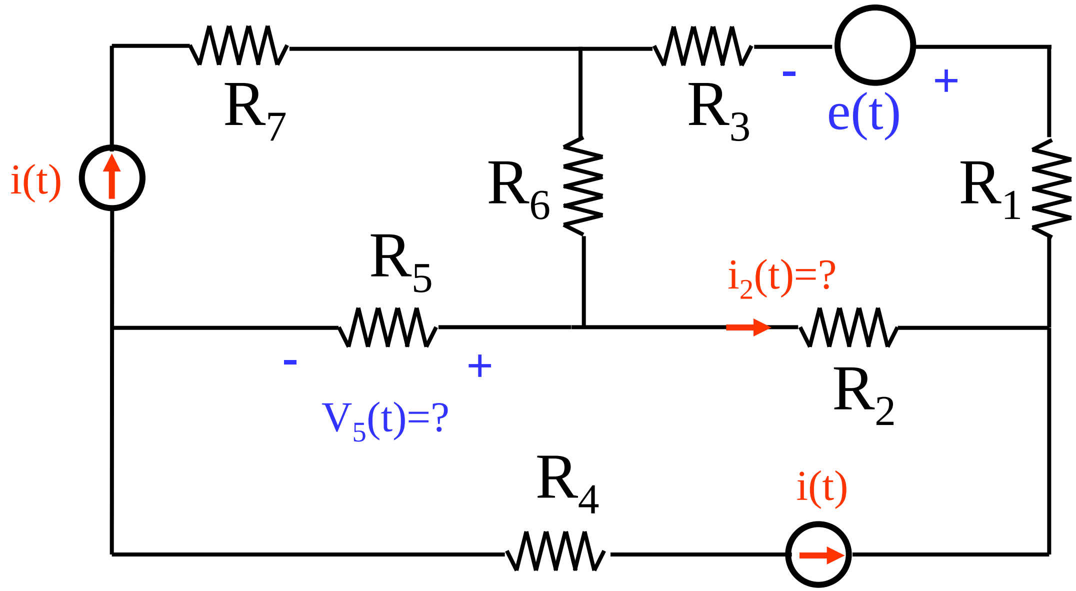
No matter how complex the circuit is, is there a way to analysis of the circuit in a systematic way?

Methods of Analysing Circuits

- **Direct Implementation of Basic Laws** (Ohms's Law and Kirchhoff's Laws together)
- **Mesh Analysis**
- **Nodal Analysis**

Direct Implementation of Basic Laws

- Most generally, an electrical circuit consists of one or more **sources** (current and voltage), which supply power to the circuit, a multiple of loops and a of junctions(nodes).



- *Known quantities usually are* the voltage of the source voltage (e(t)) and the current of the source currents (i(t)).

- *Unknown quantities usually are the currents of the voltage sources, the voltages of the current sources and the voltage ($v_5(t)$) and currents ($i_2(t)$) on the circuit elements (resistor).*

Direct Implementation of Basic Laws

The equations for finding unknown quantities can be classified:

- Kirchhoff’s Current Law (KCL) equations,
- Kirchhoff’s Voltage Law (KVL) equations,
- Current-Voltage equations of circuit elements (Ohm’s

The objective is to write equations as many as the number of unknown in the circuit

The total number of independent equations must be equal to the number of unknown quantities!

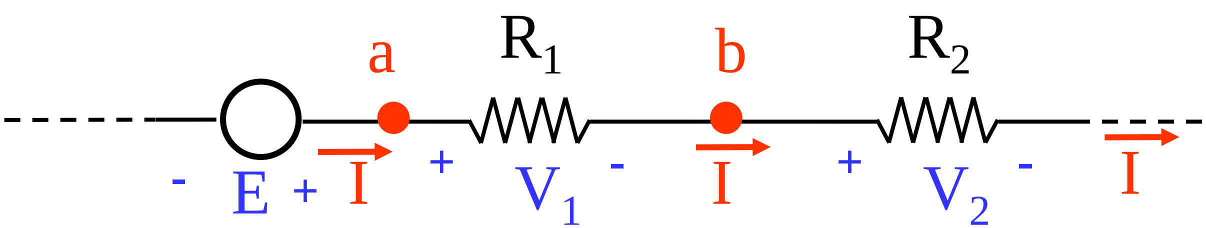
Number of the independent equations as the following:

1. The number of independent voltage-current equations of the circuit elements is equal to the number of elements
2. The number of independent KCL equations is equal to the number of one less number of junctions .
3. The number of independent KVL equations is equal to the number of independent loops (nodes) (*An independent loop is a loop with a KVL equation containing at least one unknown voltage not found in other equations*).

Steps to Follow in Circuit Analysis

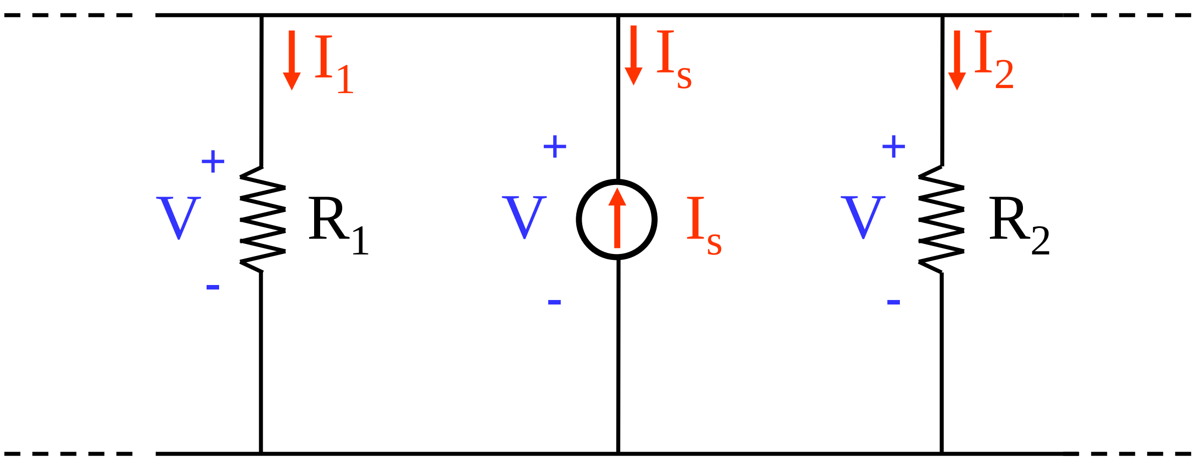
There are two circuit combinations that make it easy to identify variables:

Series Circuits



Common Current (I)

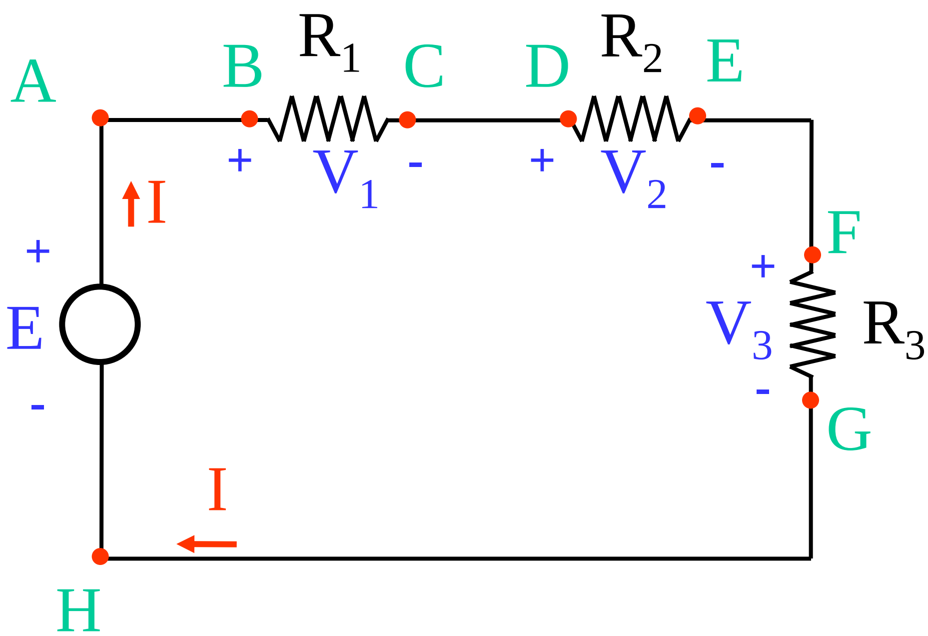
Paralle Circuits



Common Voltage (V)

Mechanical Equivalent of an Electric Circuit

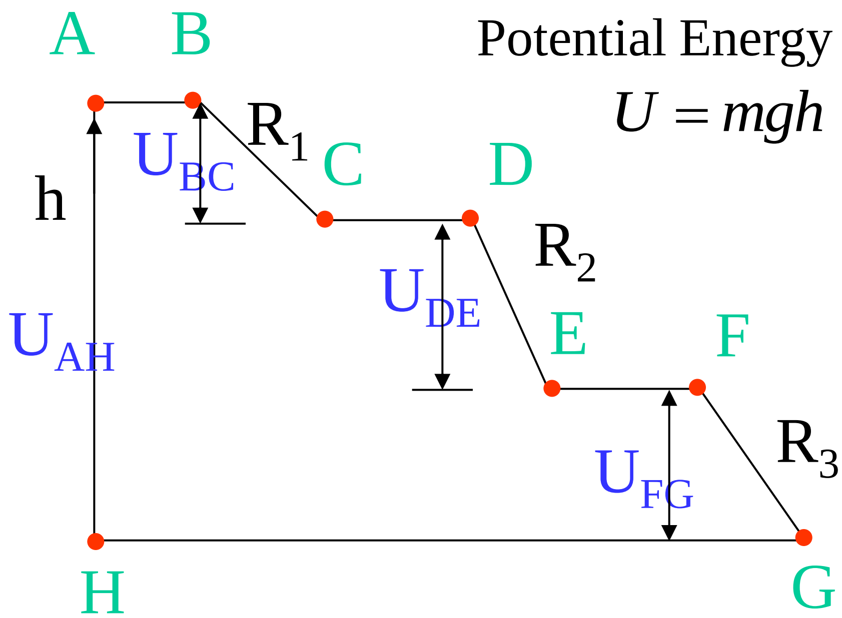
Current and Voltage



$$+E - V_1 - V_2 - V_3 = 0$$

$$E = V_1 + V_2 + V_3$$

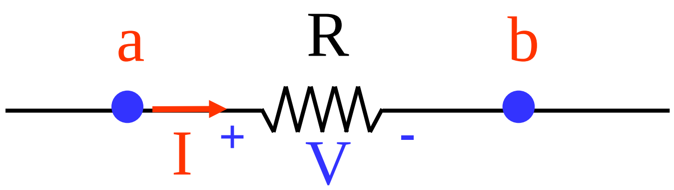
Mechanical Equivalent



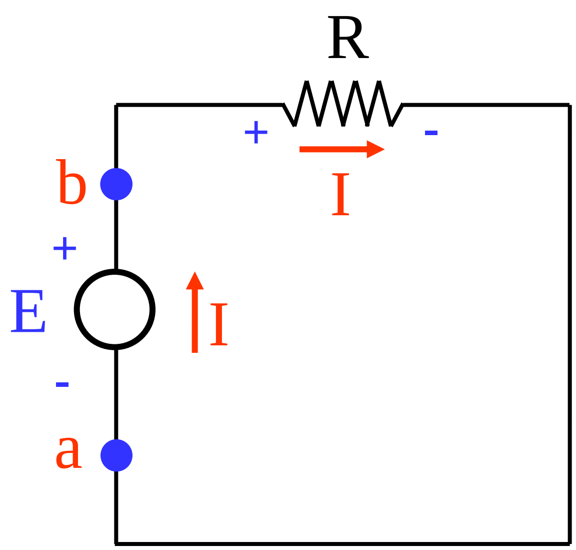
$$+U_{AH} - U_{BC} - U_{DE} - U_{FG} = 0$$

$$U_{AH} = U_{BC} + U_{DE} + U_{FG}$$

Choosing the Potential Increases and Drops



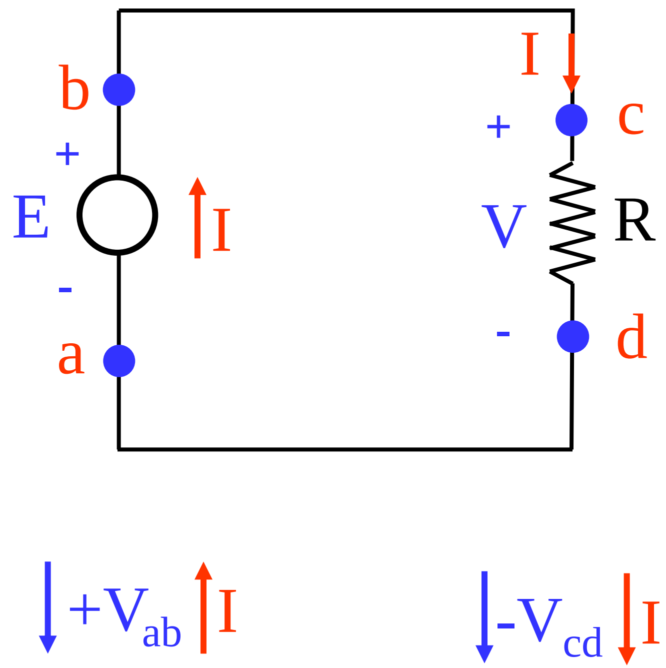
If the current is flowing from **a** to **b** via resistor R, point a has a higher potential than point b; otherwise the current does not flow through the network!



Since the voltage source provides power to the circuit, the direction of the current is from **a** (negative) to **b** (positive).

Note that this is the opposite of the above situation

Choosing the Potential Increases and Drops

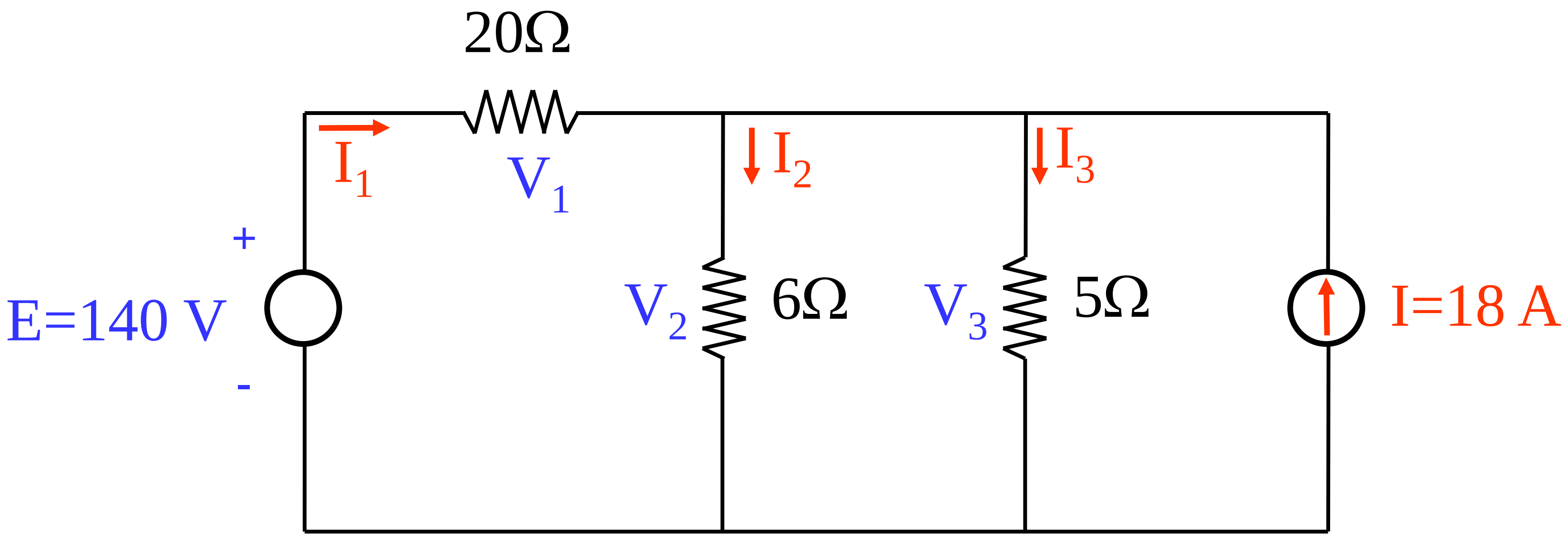


Potential rise   Potential drop

Since the voltage source provides power to the circuit, the direction of the current is from **a** (negative) to **b** (positive).

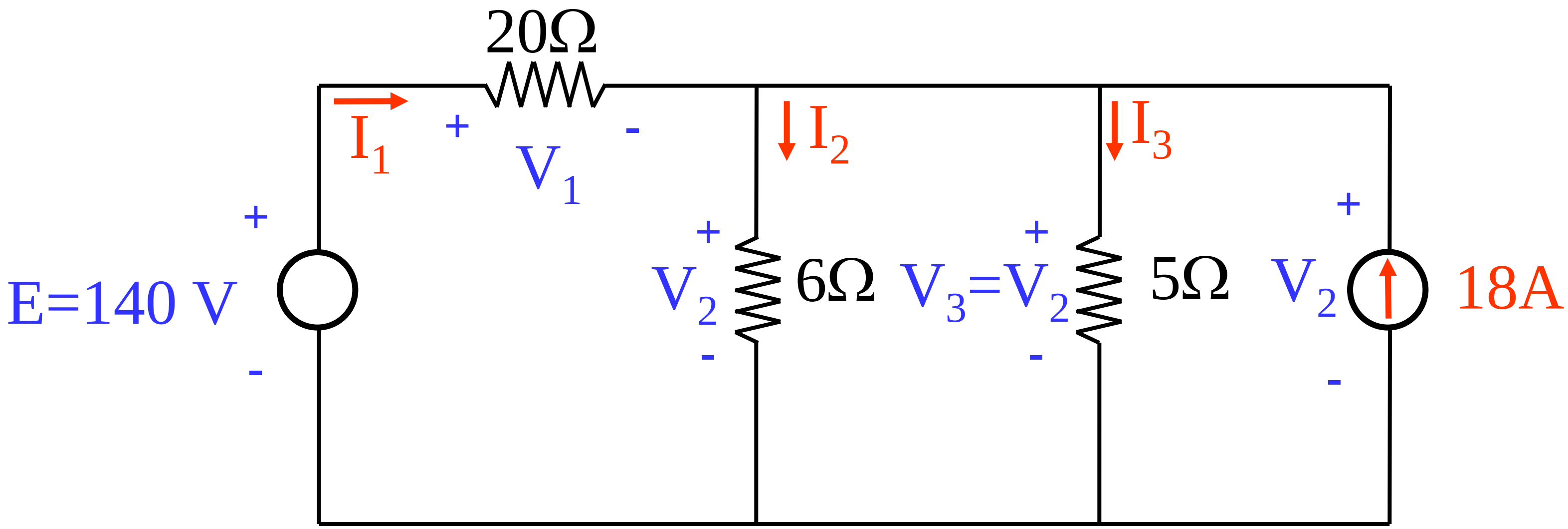
Note that current is in the same direction

Example 2.1: Find the unknown voltages (V_1, V_2 and V_3) and currents (I_1, I_2 and I_3) in the circuit below. Also, write an equation for the power balance that shows that the power supplied by the sources is equal to the power absorbed by the resistors.



Solution: The first thing to do for the solution is to determine the reference directions for unknown voltages and currents.

- Since the 140V source and the 20Ω resistor are connected in series, the current I_1 passes through both. Therefore, the voltage V_1 is as shown (the point where the current enters is positive and the point where it exits is negative).
- The resistors 6Ω, 5Ω and the 18A current source are connected in parallel, so they see a common V voltage (V_2=V_3) Accordingly, directions of currents I_2 and I_3 are determined as follows.

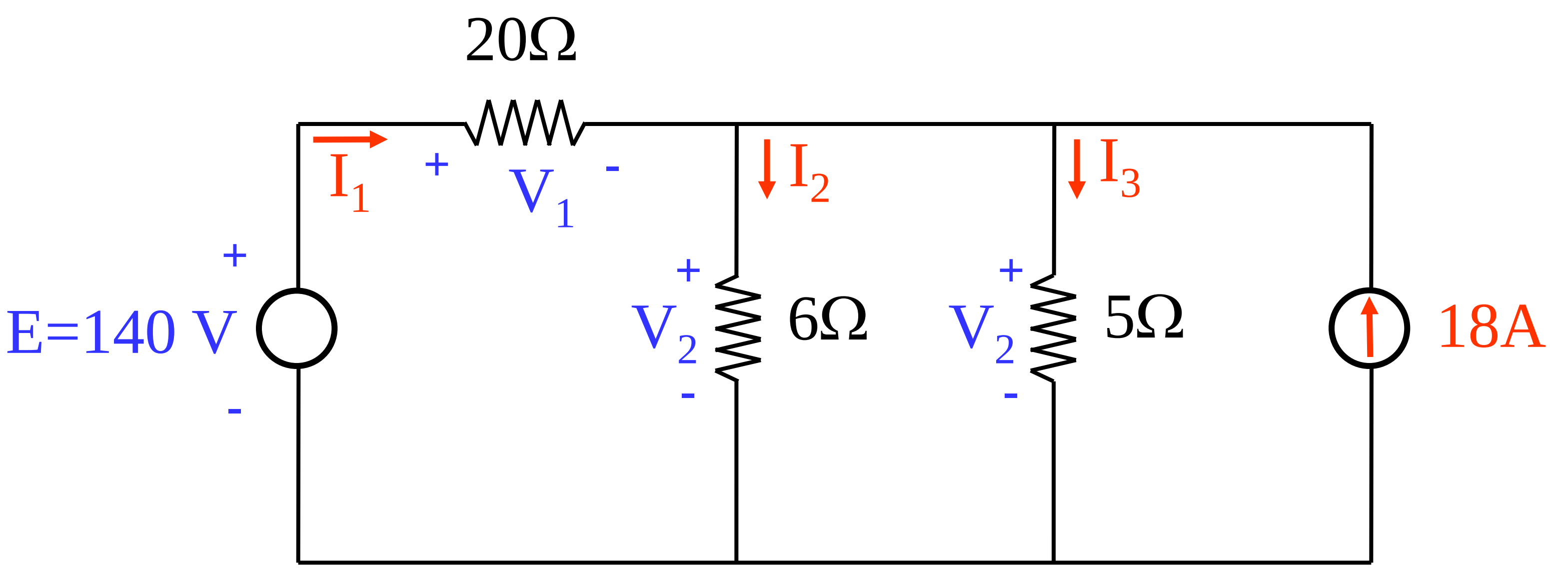


1. step: First group equations are voltage-current relations of the elements (resistors). Since there are 3 resistors in the circuit, 3 Ohm's Law equations can be written:

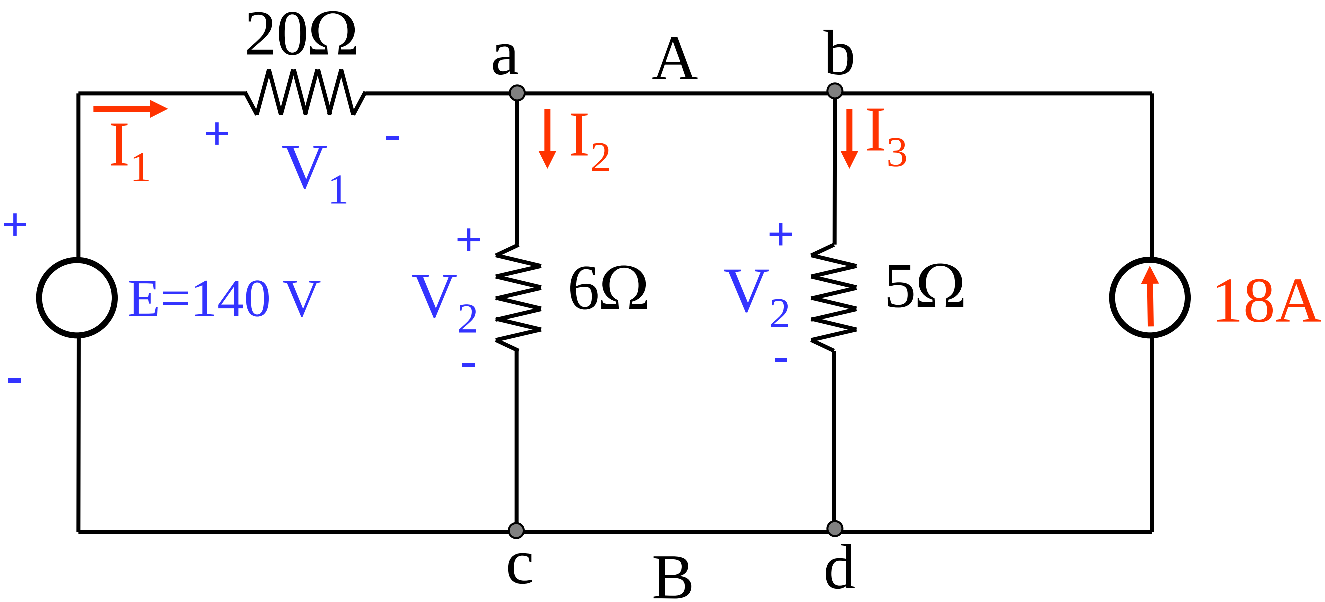
20 Ω Resistor: $V_1 = (20\Omega) I_1$ ..... (1)

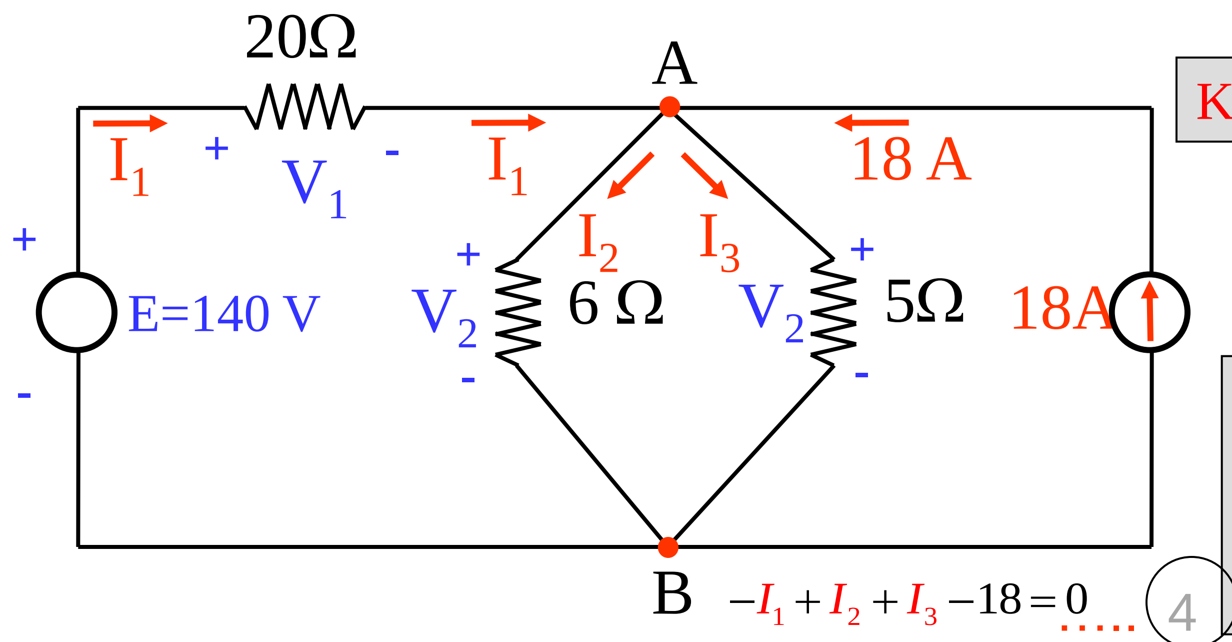
6 Ω Resistor : $V_2 = (6\Omega) I_2$ ..... (2)

5 Ω Resistor : $V_3 = (5\Omega) I_3 = V_2$ ..... (3)



2. Step: KCL equations are written. Although the number of junctions (nodes) appears to be 4 (a, b, c and d), there are actually two (ab = A and cd = B).



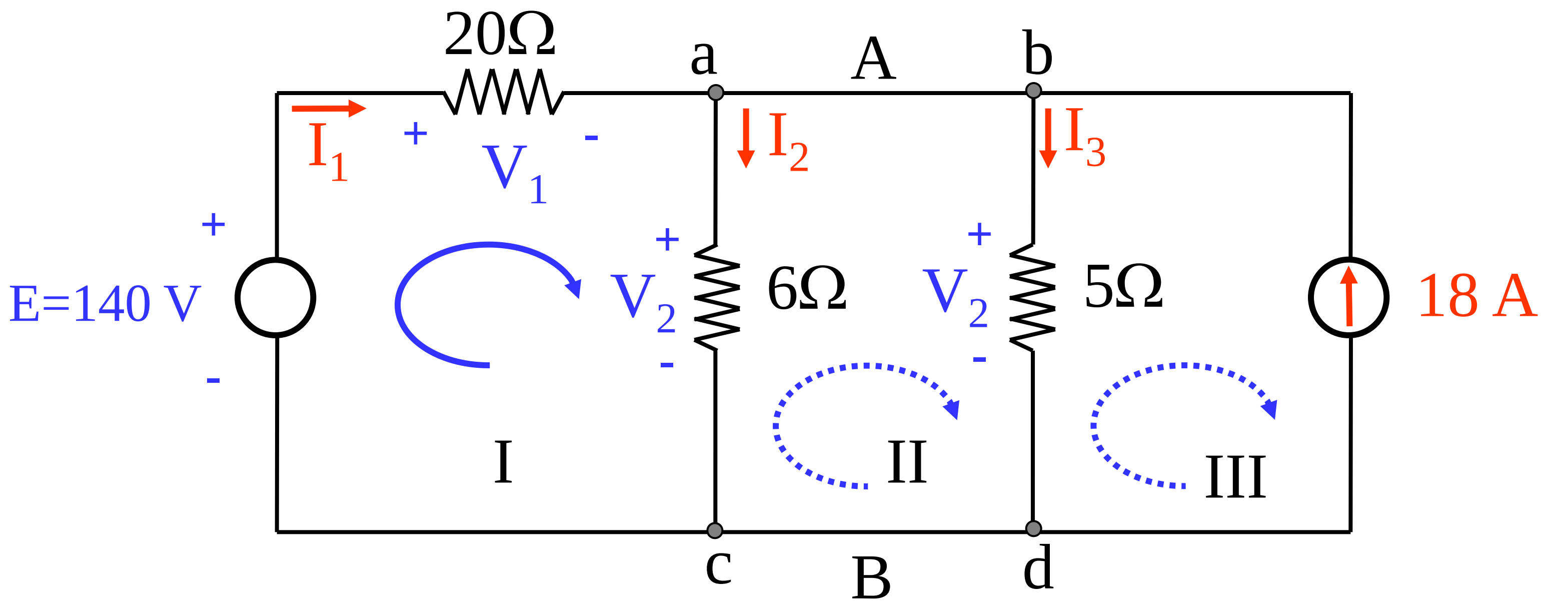


KCL for junction A:

$$I_1 - I_2 - I_3 + 18 = 0 \quad \text{..... (4)}$$

KCL for junction B will be the same as junction A (*Number of independent junction is one less*).

3. Step: KVL equations are written. The only independent loop in the circuit is the loop on the left (I) (Other loops (II and III) are not independent because they give the same unknown).



KVL equations (I): $+140 - V_1 - V_2 = 0$ ..... (5)

KVL equations (II): $+V_2 - V_2 = 0$

KVL equations (III): $+V_2 - V_2 = 0$

Same!
(Loops II and III are not independent loops!)

The solutions of the above five equations can be found by any method.

Ohm's Law (R_1)	$V_1 = (20\Omega) I_1$	..... (1)
Ohm's Law(R_2)	$V_2 = (6\Omega) I_2$	..... (2)
Ohm's Law(R_3)	$V_3 = (5\Omega) I_3$	..... (3)
KCL equations (For A):	$I_1 - I_2 - I_3 + 18 = 0$	..... (4)
KVL equations (For A):	$140 - V_1 - V_2 = 0$	..... (5)

It is usually written in either KCL or KVL equations to eliminate either current or voltage variables (equations 1-2-3 in equation 4):

$$\frac{1}{20\Omega} V_1 - \frac{1}{6\Omega} V_2 - \frac{1}{5\Omega} V_2 + 18 = 0 \quad \text{.....} \quad (6)$$

From Eqs 5 and

$$V_1 + V_2 = 140$$
$$-3V_1 + 22V_2 = 1080$$

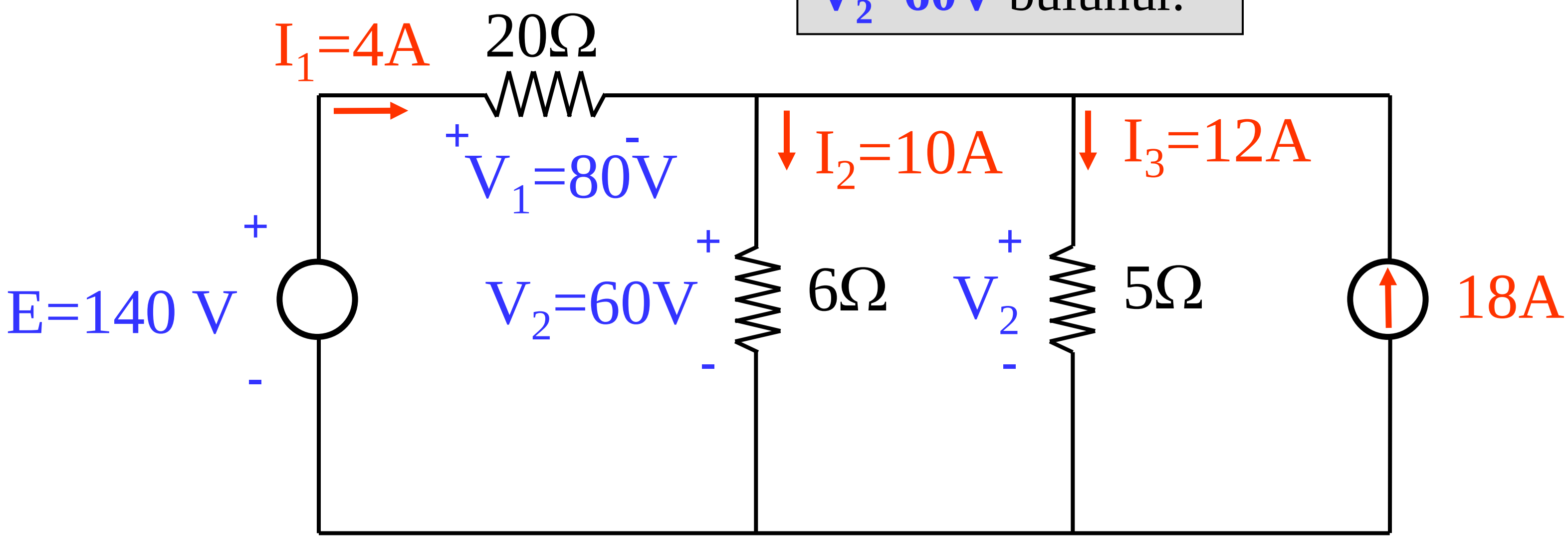
⇒ **V_1=80V**; **V_2=60V** found

Akım denklemlerinden (1-3), akımlar **I_1=4A**, **I_2=10A** ve **I_3=12A** bulunur.

Write a statement for the power balance indicating that the power supplied by the sources is equal to the power absorbed by the resistors.

Currents: **I_1=4A**, **I_2=10A** and **I_3=12A** can be found

V_1=80V;
V_2=60V bulunur.



Power Supplied = Power Consumed

Power balance can be calculated as follows:

Power supplied to the circuit:

Voltage Source: $P=E.I=(140\ V).(4\ A) = 560\ W$
Current Source: $P=E.I=(60\ V).(18\ A) = 1080\ W$

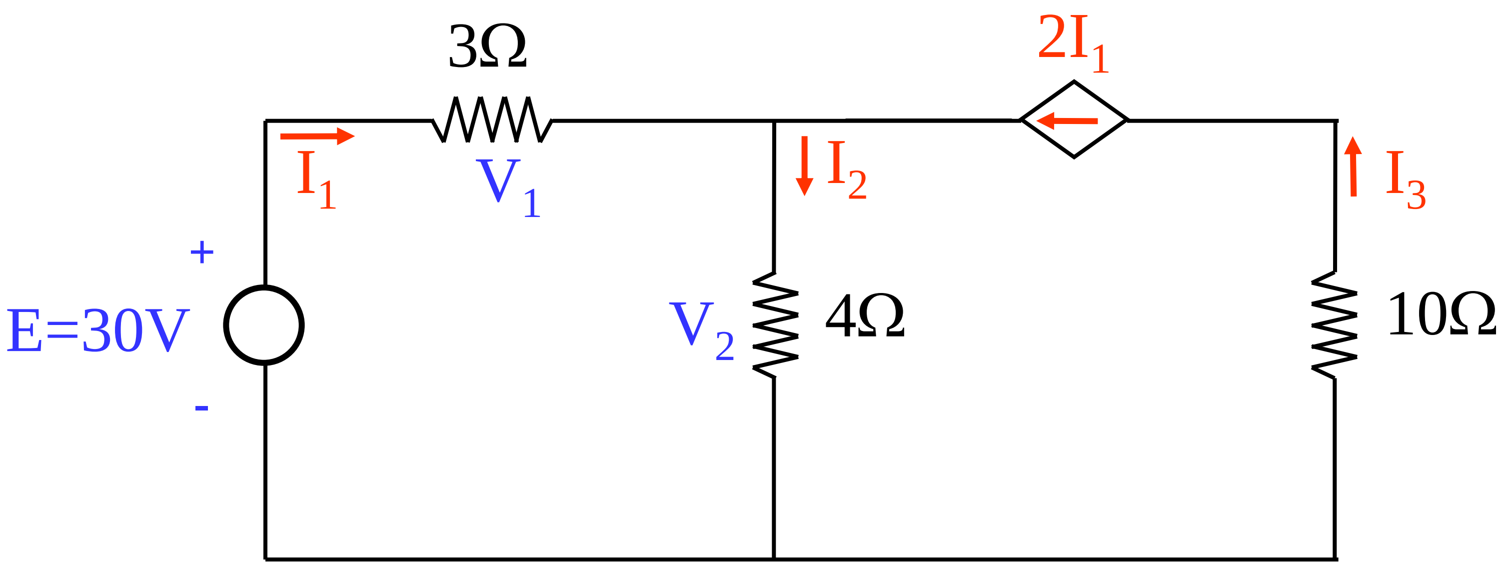
Total: 1640 W

Power Consumed in the circuit:

5 Ω Resistor: $P=RI^2=(5\ \Omega).(12\ A)^2 = 720\ W$
6 Ω Resistor: $P=RI^2=(6\ \Omega).(10\ A)^2 = 600\ W$
20 Ω Resistor: $P=RI^2=(20\ \Omega).(4\ A)^2 = 320\ W$

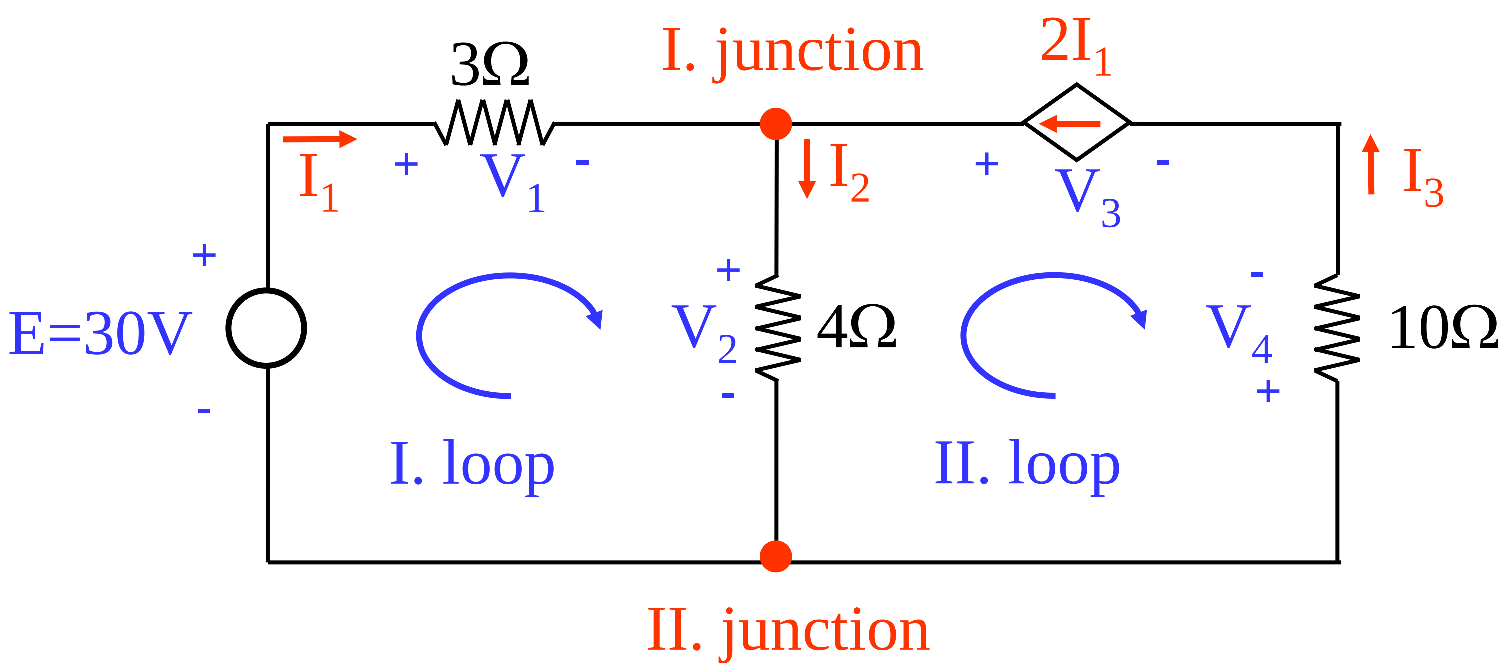
Total: 1640 W

Example-2.2: The following circuit contains a 30V constant voltage source on the left loop and a current dependent current source (supplying current proportional with I_1) is connected to the current on the right loop. Find unknown voltages (V_1 and V_2) and currents (I_1, I_2 and I_3)



Solution: The first step for the solution is to determine the reference directions for unknown voltages and currents.

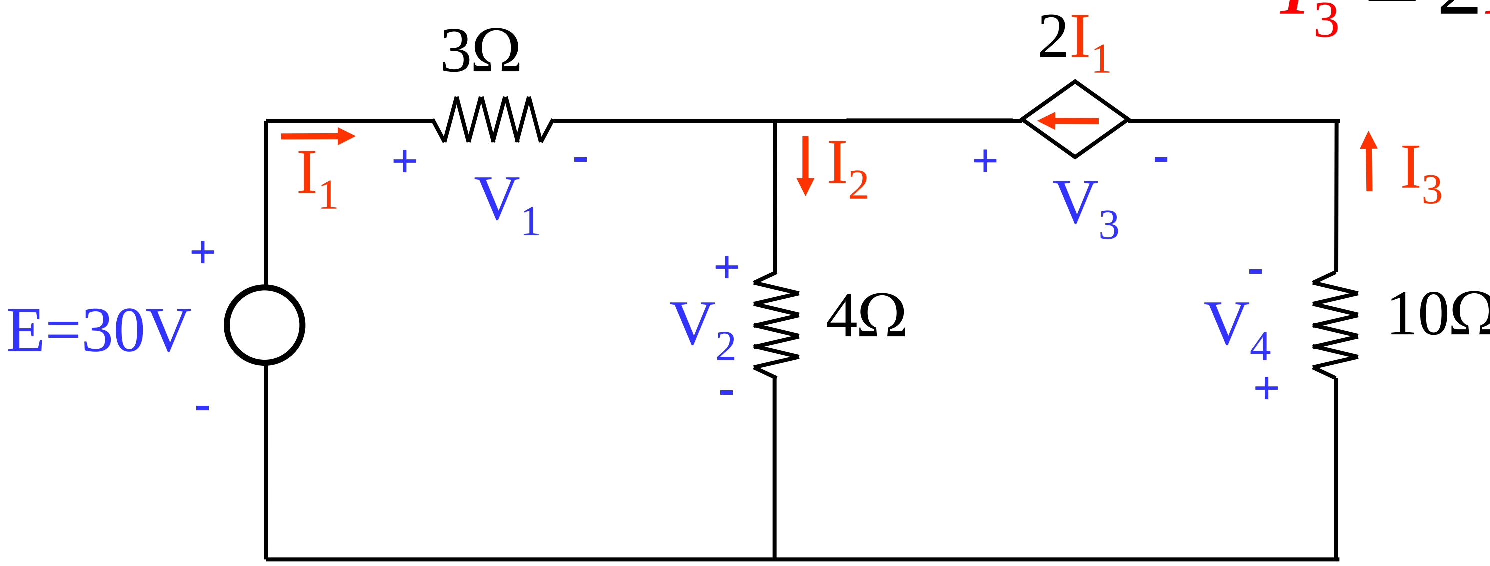
- The circuit has 3 resistors, **two junctions** and **two independent loops**.
- Three Ohm's Law equations, one KCL and two KVL equations can be written.
- Although I_1 and $2I_1$ currents are unknown for now, we can take reference directions as follows:



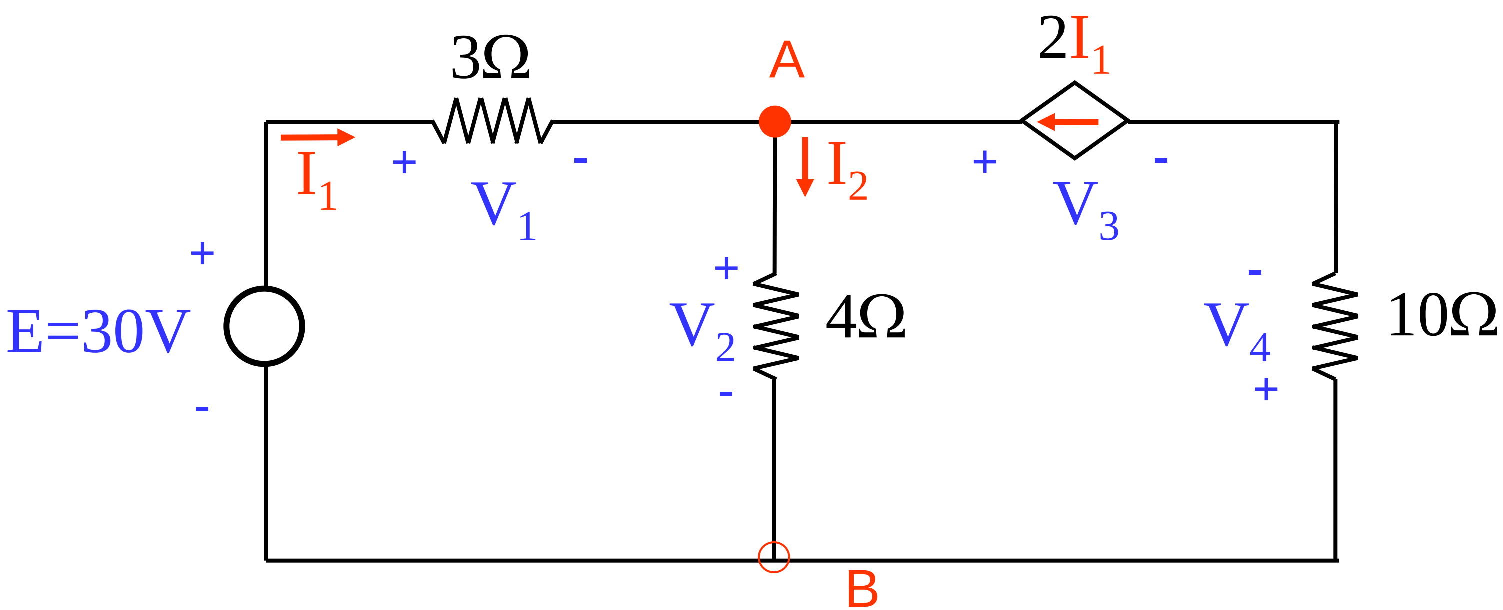
Step 1: First Group equations, there are 3 Ohm's Law equations:

Ohm’s Law (R_1)	$V_1 = (3\Omega) I_1$	 1
Ohm’s Law (R_2)	$V_2 = (4\Omega) I_2$	 2
Ohm’s Law (R_3)	$V_4 = (10\Omega)(2I_1)$	 3

$$I_3 = 2I_1$$



Step 2: KCL equations are written. The number of independent junctions is one (junction A)

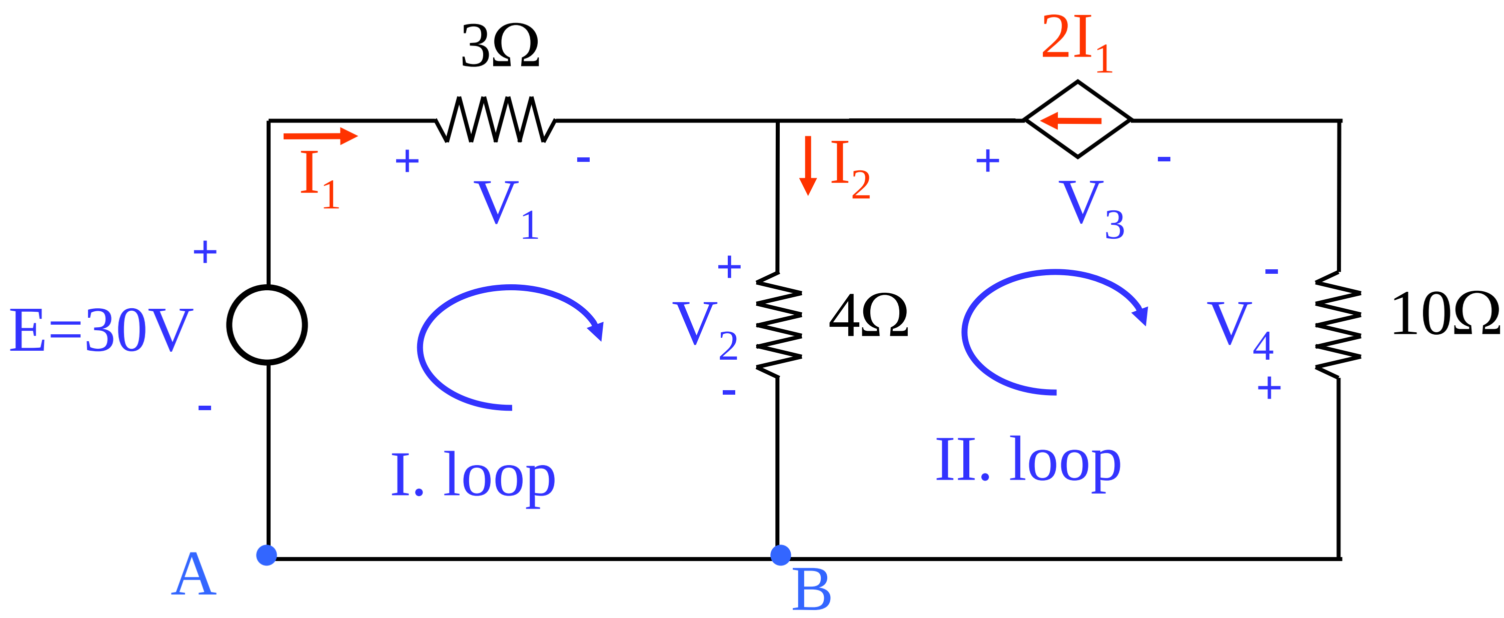


KCL for junction A: $I_1 - I_2 + 2I_1 = 0$ ..... (4)

KCL for junction B: $-I_1 + I_2 - 2I_1 = 0$ ..... (4')

Equation 4 and 4' are same!

Step 3: Write the KVL equations. KVL are written around I and II loops



KVL for loop I:
(Starting from A to point A)

$$30 - V_1 - V_2 = 0 \quad \text{.....} \; (5)$$

KVL for loop II:
(Starting from B to point B)

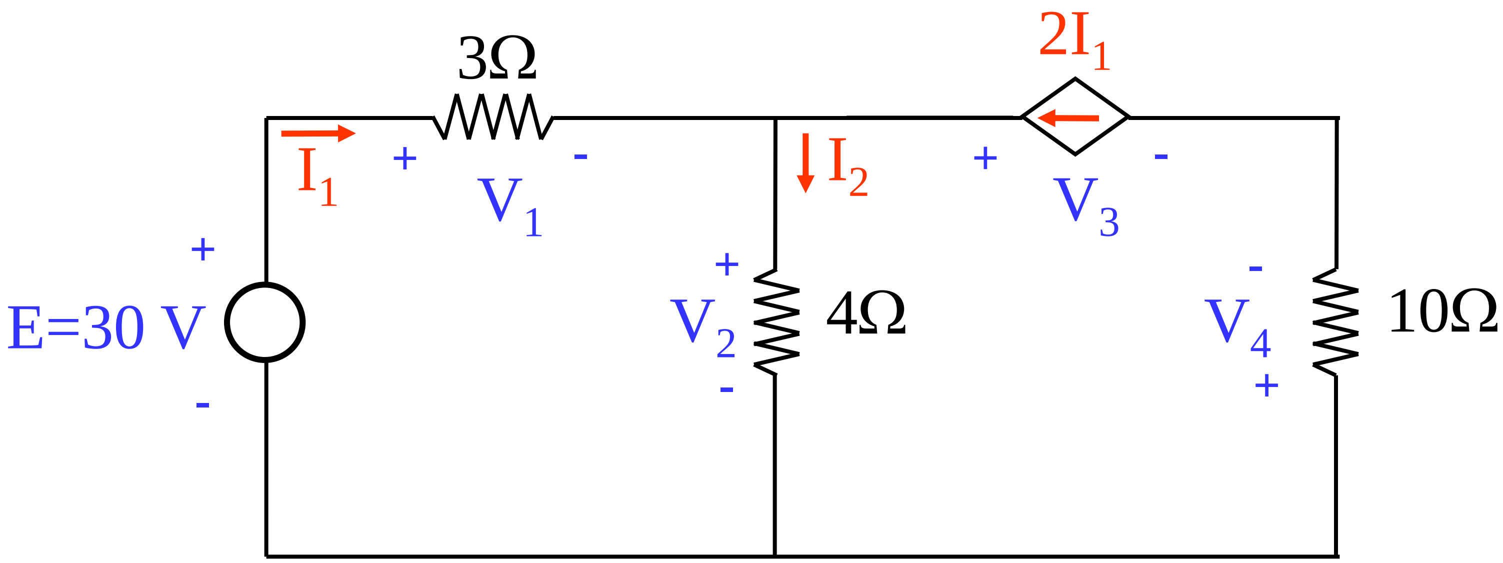
$$V_2 - V_3 + V_4 = 0 \quad \text{.....} \; (6)$$

The solutions of the above five equations can be found by any method.

From current eqs. (1-3), the currents **I_1=2A**, **$2I_1$=4A** and **I_2=6A** can be found

From eqs. 5 and 6:

V_2=24V
V_3=64V
V_4=40V can be found

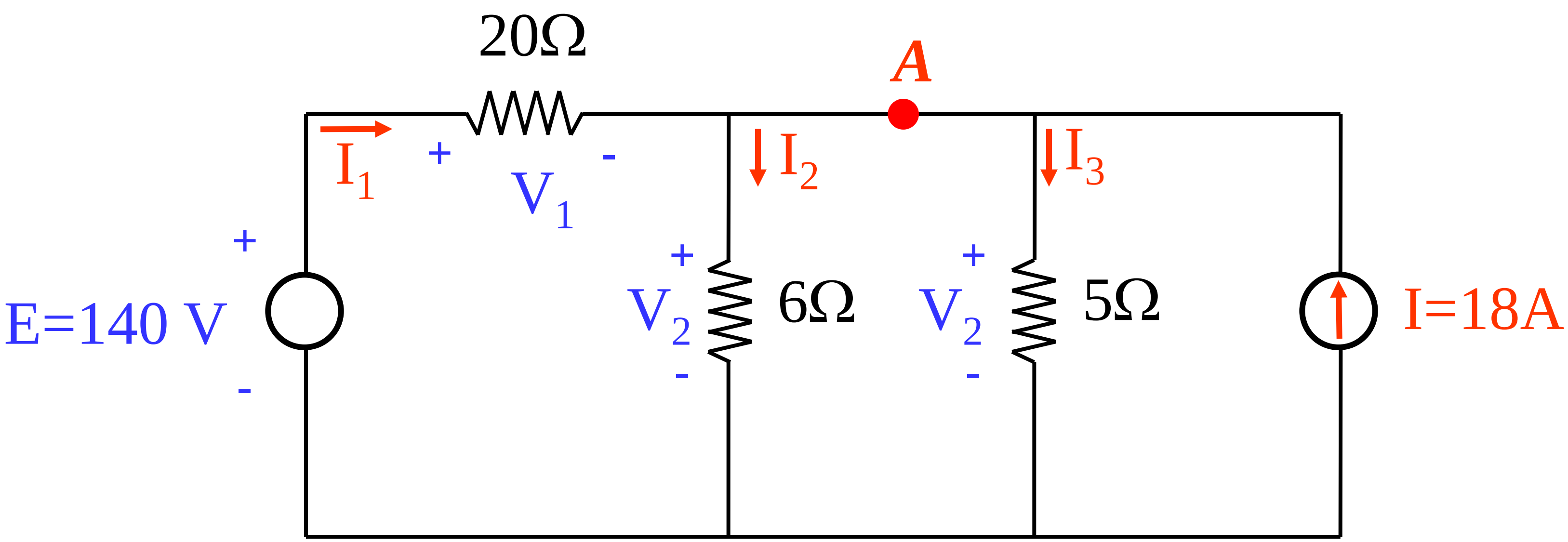


Comment: Example-2.1 and Example-2.2 illustrate the application of the method of Direct Implementation of Basic Laws in the easiest and clearest manner.

The obtained equations can be made more compact by two simplification.

1- Defining the current variable in terms of the voltage variable (or vice versa): such a simplification will allow it to be written without the need to explicitly write Ohm's Law. Using KCL for junction A:

$$I_1 - I_2 - I_3 + 18 = 0 \quad \ldots\ldots (4) \quad \Rightarrow \quad \frac{1}{20\Omega}V_1 - \frac{1}{6\Omega}V_2 - \frac{1}{5\Omega}V_2 + 18 = 0 \quad \ldots\ldots (6)$$



2- In the second simplification, the need to write either KCL equations or KVL equations is avoided by selecting the variables in terms of other previously selected variables.

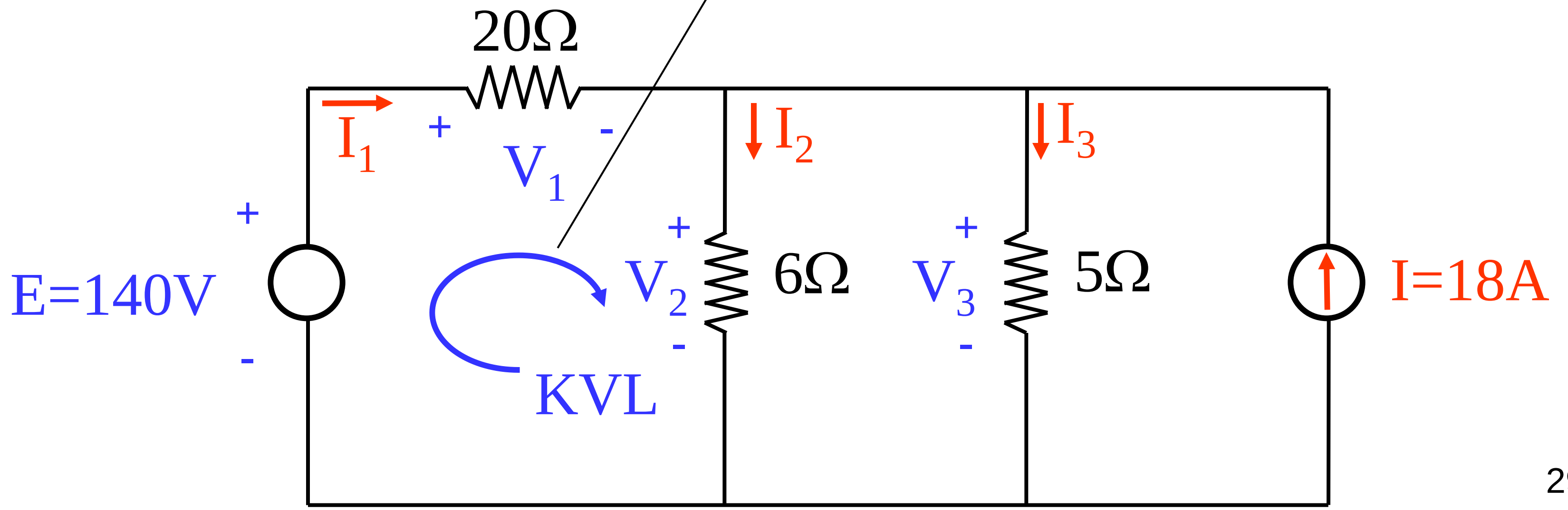
For example, in Example-2.1, the voltage V_1 is $140-V_2$ (from KVL).

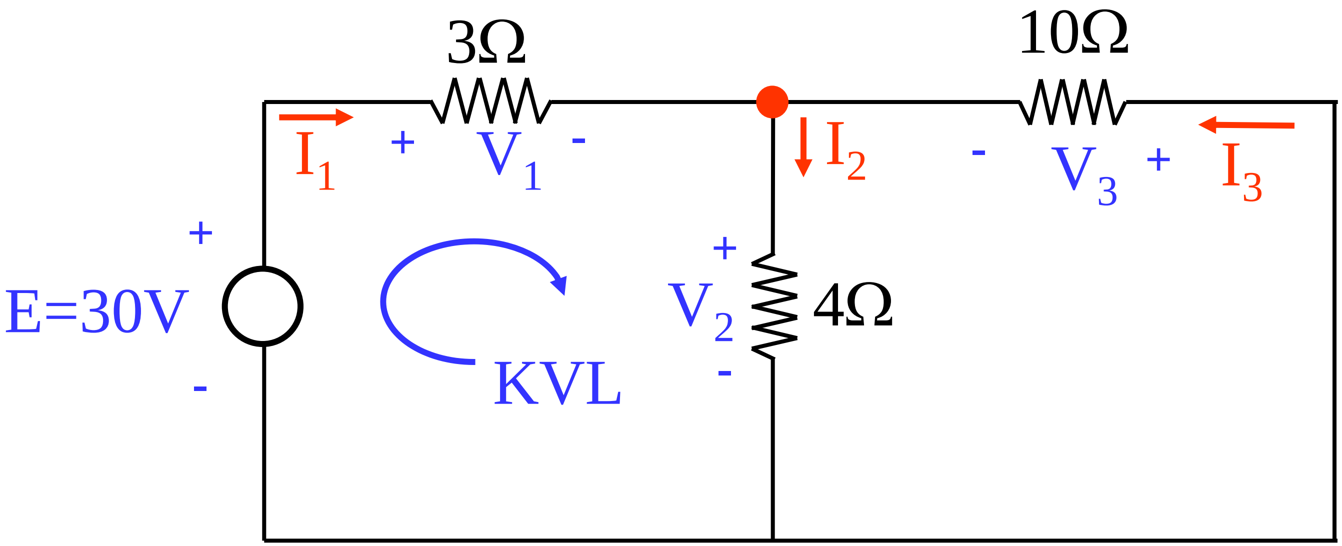
$$140 - V_1 - V_2 = 0 \quad \ldots\ldots (5)$$

$$V_1 = 140 - V_2$$

$$\frac{1}{20}V_1 - \frac{1}{6}V_2 - \frac{1}{5}V_2 + 18 = 0 \quad \ldots\ldots (6)$$

$$\frac{1}{20\Omega}(\overbrace{140 - V_2}^{V_1}) + \frac{1}{6\Omega}V_2 + \frac{1}{5\Omega}V_2 + 18 = 0$$





KCL: $+I_1 - I_2 + I_3 = 0$

$$+\frac{1}{3\Omega}V_1 - \frac{1}{4\Omega}V_2 + \frac{1}{10\Omega}V_3 = 0$$

$$30 - V_1 - V_2 = 0$$

$$V_1 = 30 - V_2$$

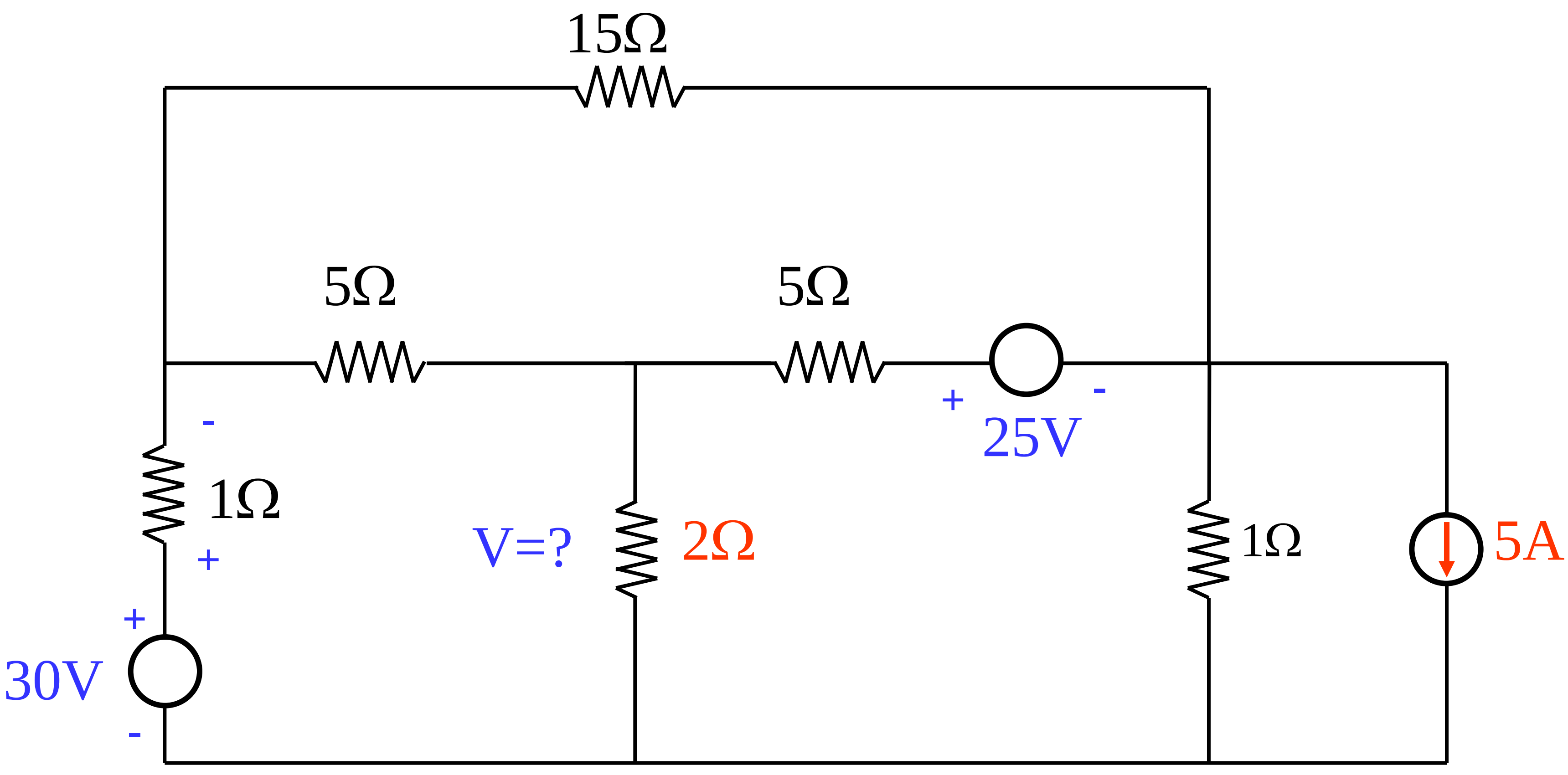
KCL in terms of KVL : $\frac{1}{3\Omega}(30 - V_2) - \frac{1}{4\Omega}V_2 + \frac{1}{10\Omega}V_3 = 0$

$$+V_2 + V_3 = 0$$

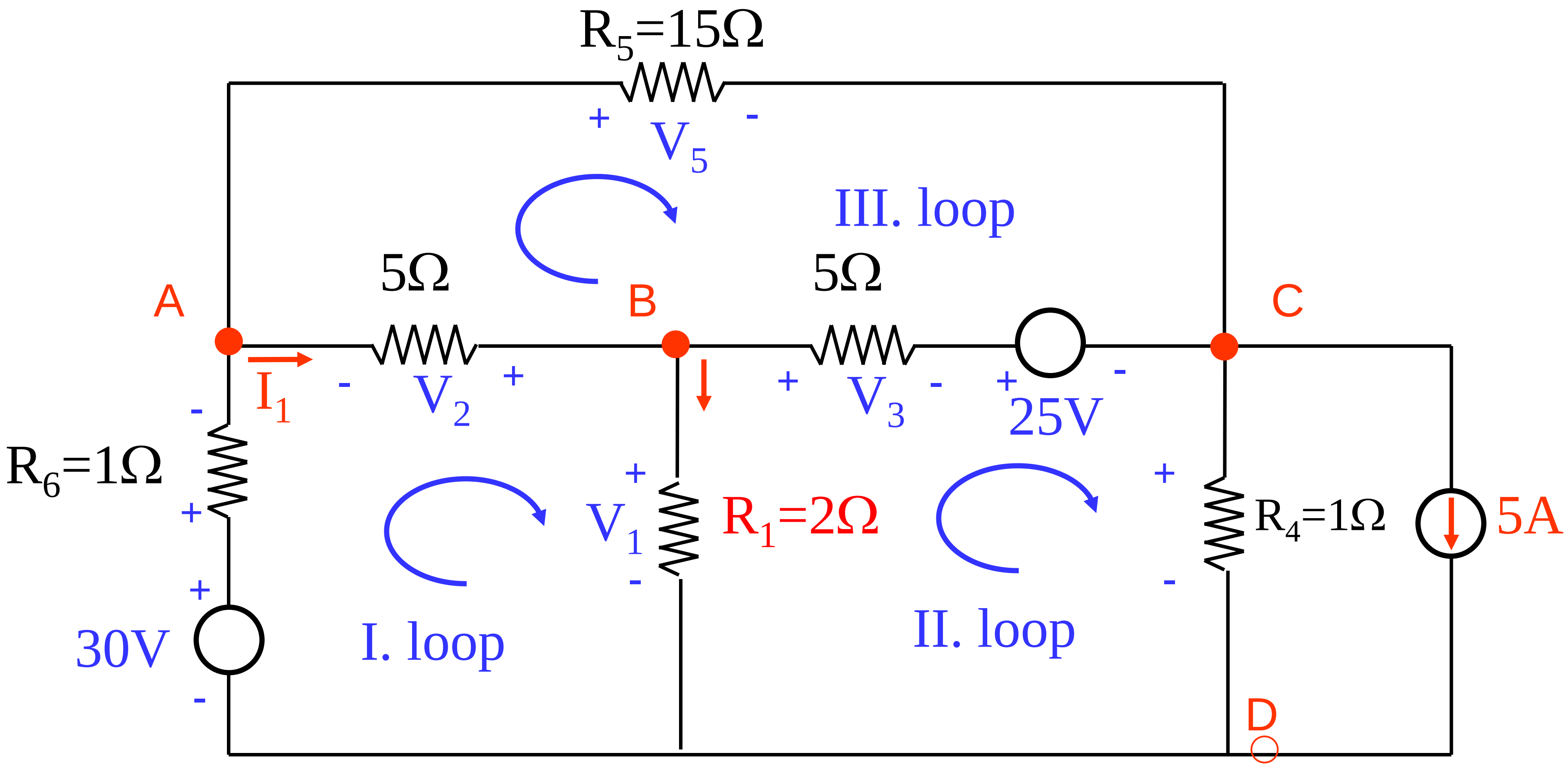
$$V_3 = -V_2$$

$$\frac{1}{3\Omega}(30 - V_2) - \frac{1}{4\Omega}V_2 - \frac{1}{10\Omega}V_2 = 0$$

Example-2.3: Find the voltage between the terminals of the 2Ω resistor in the circuit below. To facilitate the solution, specify all currents in terms of voltage variables and use the KVL equations when selecting variables.



Since the voltage on the 2Ω resistor is requested, let's show it as V_1 and the other unknowns as V_2 and V_3. Other voltages can be expressed in terms of these voltages.



Voltage between resistor R_6 (I. loop KVL):

$$V_{6\Omega} = +V_2 - V_1 + 30 \quad ..... \quad (1)$$

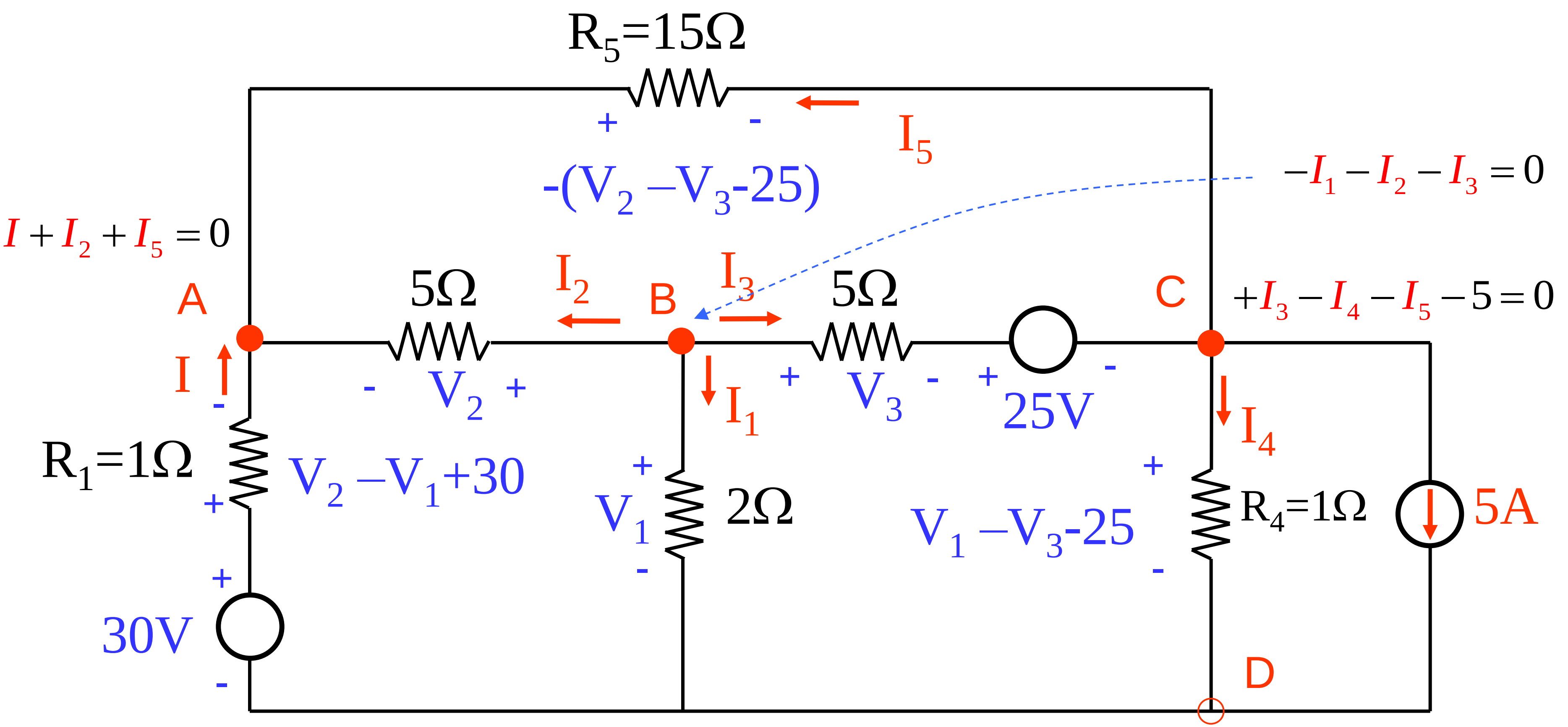
Voltage between resistor R_4 (II. loop KVL):

$$V_{4\Omega} = +V_1 - V_3 - 25 \quad ..... \quad (2)$$

Voltage between resistor R_5 (III. loop KVL):

$$V_{5\Omega} = +25 + V_3 - V_2 \quad ..... \quad (3)$$

2. Step: Currents at intersections must be written. There are 4 junctions (A, B, C and D), so **3 independent equations** can be written.



KCL for junction A:

$$\frac{1}{1\Omega}(V_2 - V_1 + 30) + \frac{1}{5\Omega}V_2 + \frac{1}{15\Omega}(V_2 - V_3 - 25) = 0 \quad \ldots\ldots (4)$$

KCL for junction B:

$$-\frac{1}{5\Omega}V_2 - \frac{1}{2\Omega}V_1 - \frac{1}{5\Omega}V_3 = 0 \quad \ldots\ldots (5)$$

KCL for junction C:

$$-\frac{1}{15\Omega}(-V_2 + V_3 + 25) + \frac{1}{5\Omega}V_3 - \frac{1}{1\Omega}(V_1 - V_3 - 25) - 5 = 0 \quad .. (6)$$

If we multiply denominator of the equations by 30

$$-30V_1 + 38V_2 - 2V_3 = -850$$
$$-15V_1 - 6V_2 - 6V_3 = 0$$
$$-30V_1 - 2V_2 + 38V_3 = -650$$

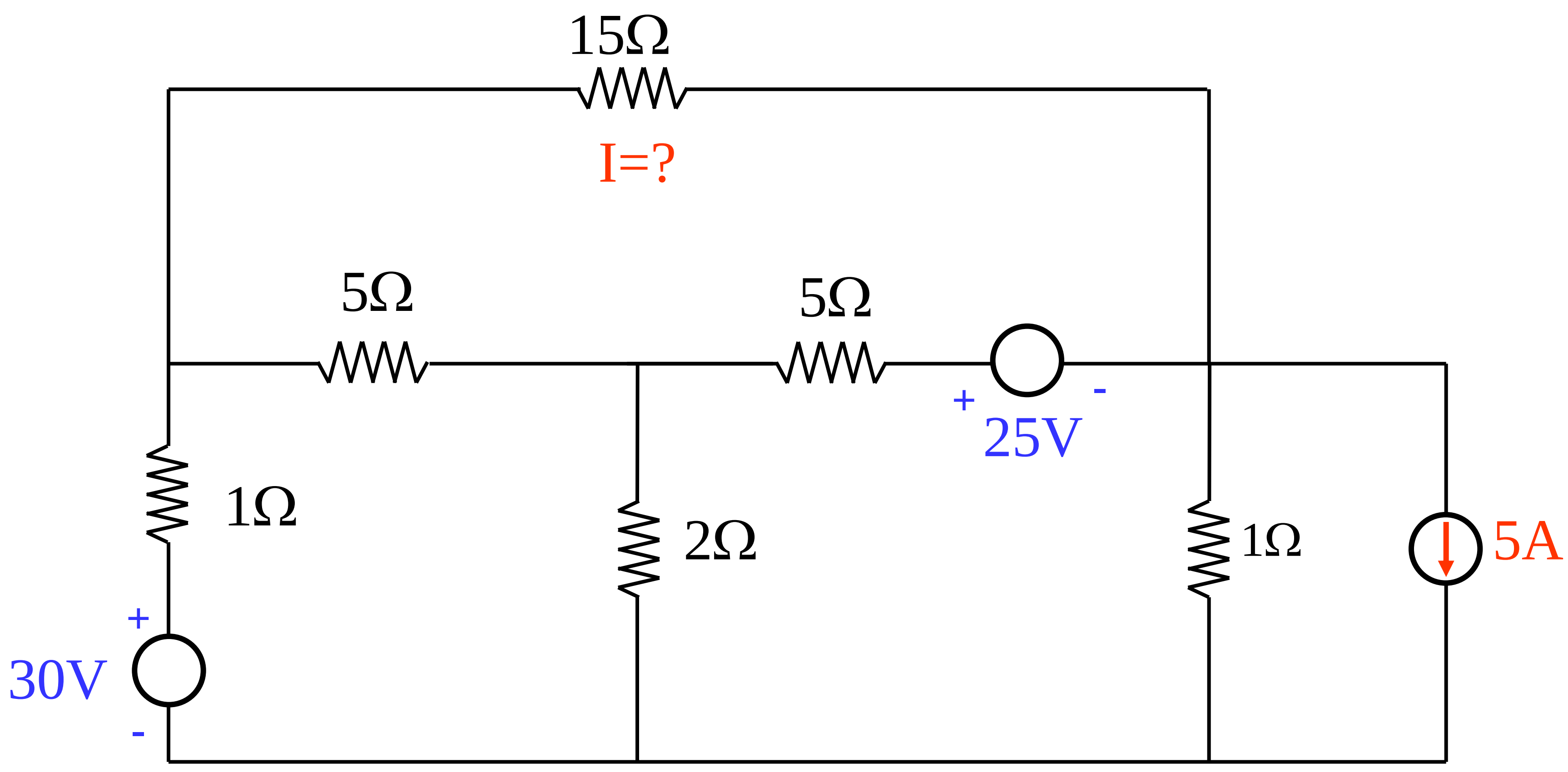
Common solution of multivariable equations can be done by using determinants and Cramer Rule:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

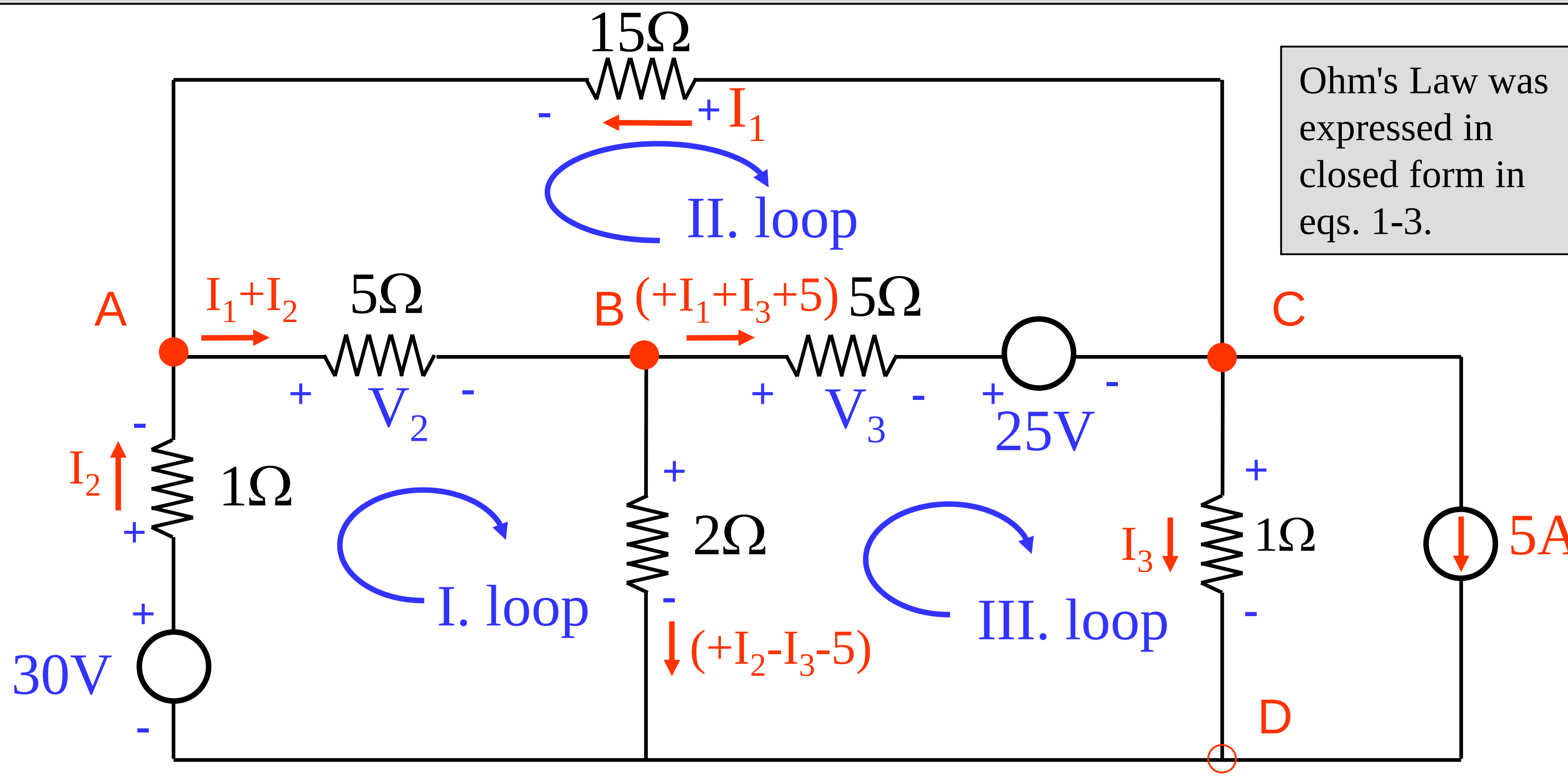
V_1 potential:

$$V_1 = \frac{\begin{vmatrix} -850 & 38 & -2 \\ 0 & -6 & -6 \\ -650 & -2 & 38 \end{vmatrix}}{\begin{vmatrix} -30 & 38 & -2 \\ -15 & -6 & -6 \\ -30 & -2 & 38 \end{vmatrix}} = \frac{-360000}{-36000} = 10\ V$$

Example-2.4: Calculate the current through the 15Ω resistor in the circuit below. To simplify the solution, specify all voltages in terms of current variables and use KCL equations when selecting variables.



Solution: Let us express all voltages in terms of current variables and use KCL equations in the selection of variables. First, we define currents I_1, I_2 and I_3. Other resistors can be found in the KCL equations.



Ohm's Law was expressed in closed form in eqs. 1-3.

Loop I: $+30 - V^{1\Omega} - V_2 - V^{2\Omega} = 0$

$$+30 - (1\Omega)I_2 - (5\Omega)(I_1 + I_2) - (2\Omega)(I_2 - I_3 - 5) = 0 \ldots (1)$$

Loop II: $+V^{15\Omega} + 25 + V^{5\Omega} + V^{5\Omega} = 0$

$$+(15\Omega)I_1 + 25 + (5\Omega)(I_1 + I_3 + 5) + (5\Omega)(I_1 + I_2) = 0 \ldots (2)$$

Loop III: $+V^{2\Omega} - V^{5\Omega} - 25 - V^{1\Omega} = 0$

$$+(2\Omega)(I_2 - I_3 - 5) - (5\Omega)(I_1 + I_3 + 5) - 25 - I_3 = 0 \ldots (3)$$

System of equations giving the currents (three equations three unknowns):

$$5I_1 + 8I_2 - 2I_3 = 40$$
$$-25I_1 - 5I_2 - 5I_3 = 50$$
$$5I_1 - 2I_2 + 8I_3 = -60$$

KCL for junction A:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

From Cramer's Rule, the current I_1:

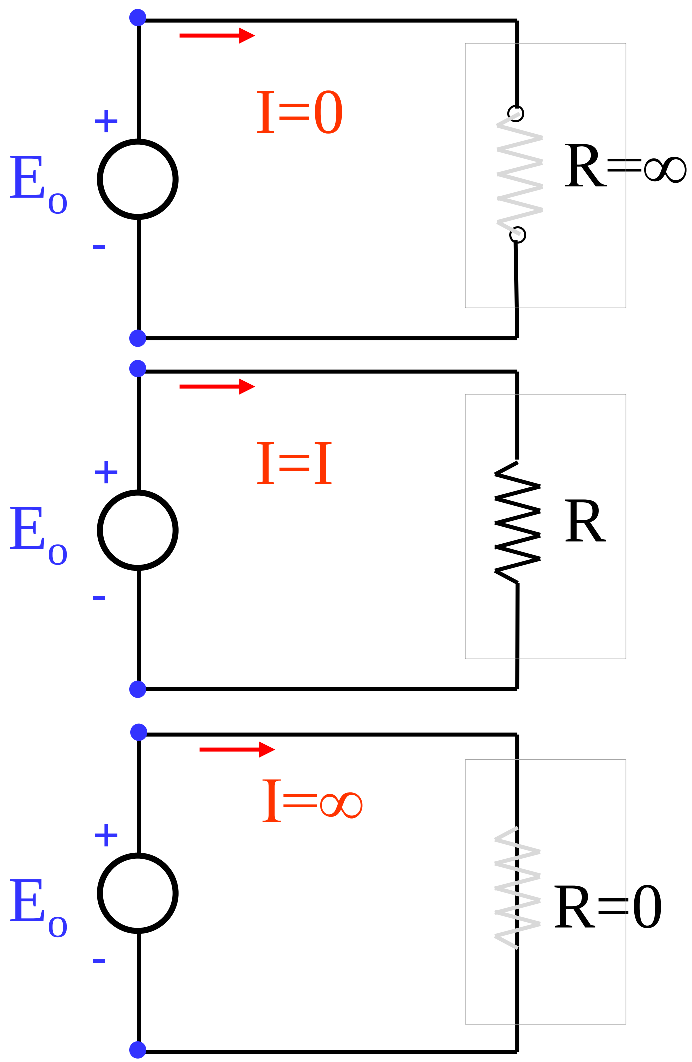
$$I_1 = \frac{\begin{vmatrix} 40 & 8 & -2 \\ 50 & -5 & -5 \\ -60 & -2 & 8 \end{vmatrix}}{\begin{vmatrix} 5 & 8 & -2 \\ -25 & -5 & -5 \\ 5 & -2 & 8 \end{vmatrix}} = \frac{-2000}{1000} = -2\ A$$

The minus sign indicates that the current I_1 is in the opposite direction to the selected direction.

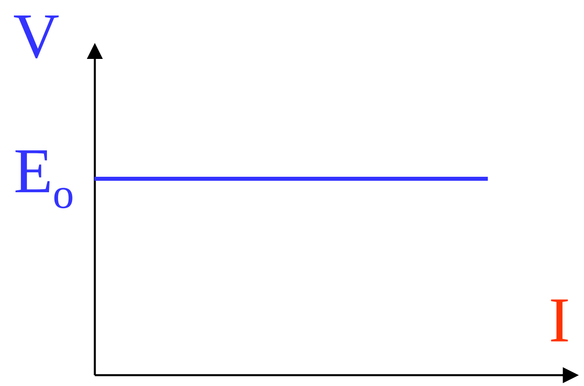
Real Voltage Sources and Source Conversion

Power sources used may approach the ideal, but never be ideal!

Ideal Voltage Source



Not Realistic!



An ideal voltage source tries to keep the voltage constant even if the load R is zero, which is unrealistic.

$$E_o = RI$$

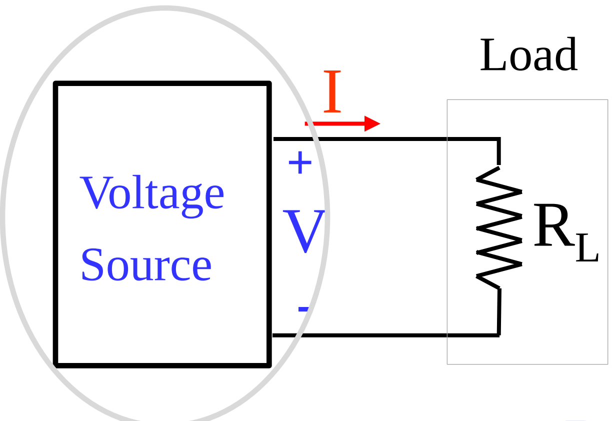
$$R \to 0 \quad E_o = constant = 0.\infty$$

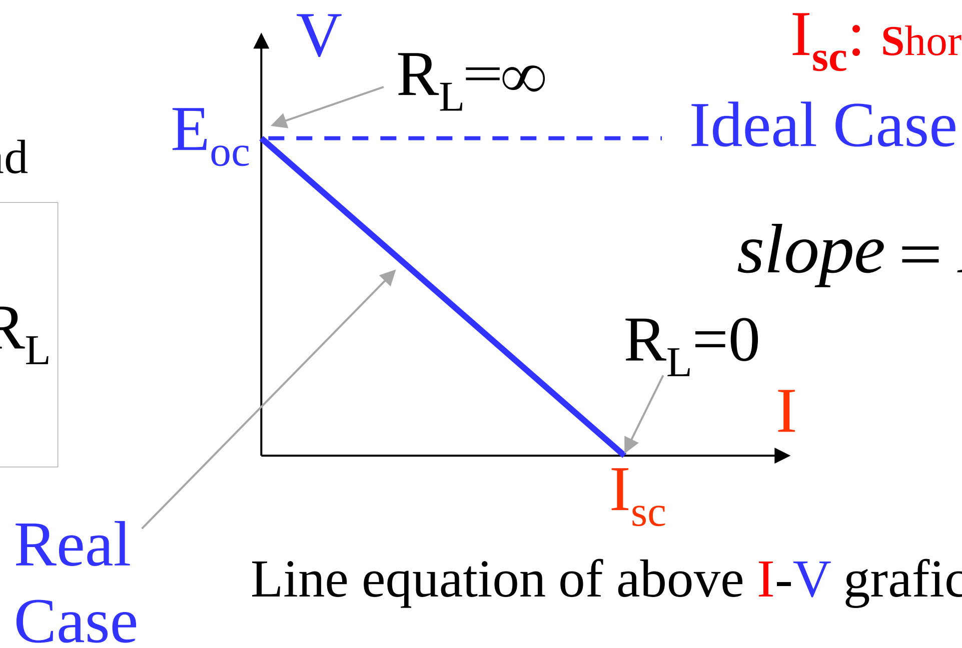
Real Voltage Source and Source Conversion

Real (Voltage) source and its I-V curve:

E_{oc}: **Open Circuit voltage**

I_{sc}: **Short Circuit current**



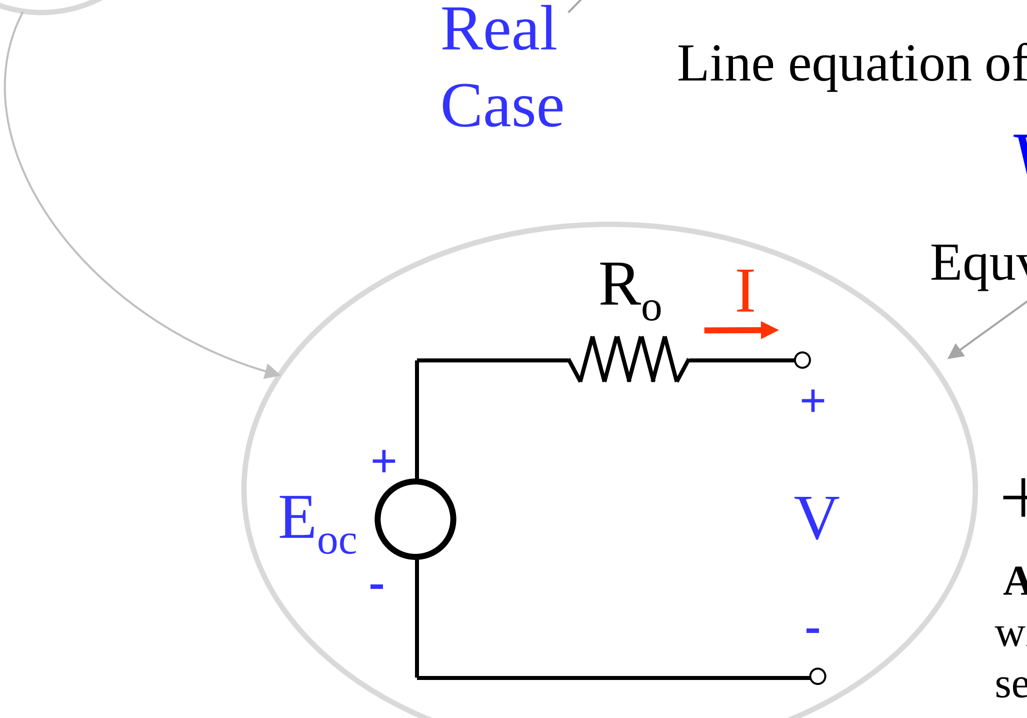


$$slope = R_o = -\frac{E_{oc}}{I_{sc}}$$

Line equation of above I-V grafic:

$$V = E_{oc} - R_o I$$

Equvalent circuit for above equation:



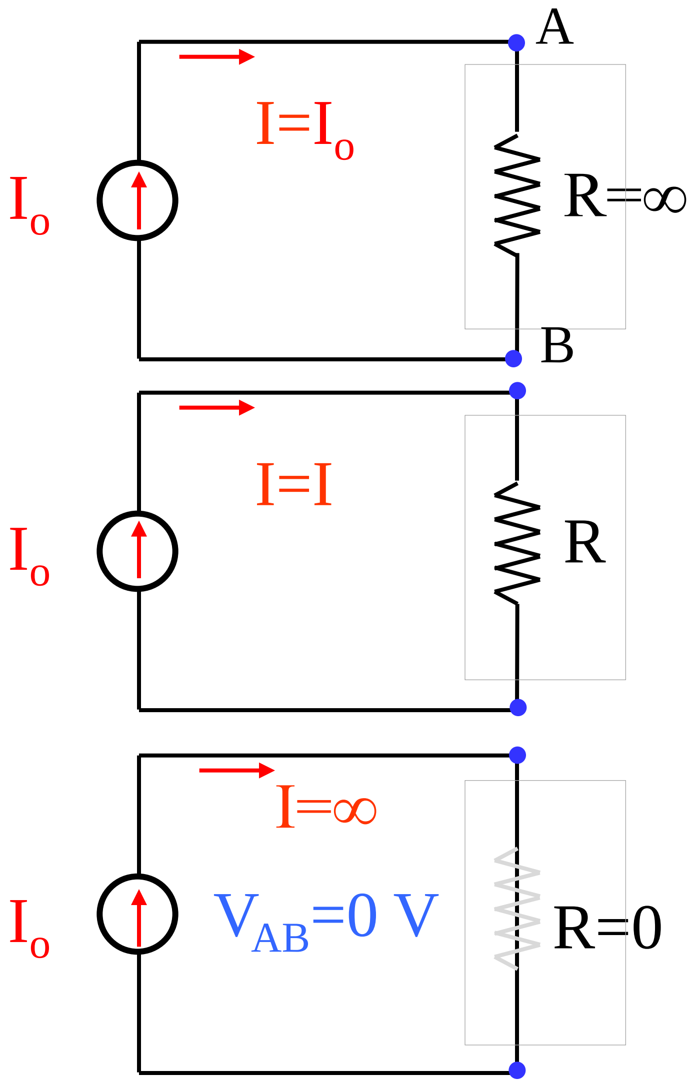
$$+E_{oc} - R_o I = V \quad \ldots\ldots (1)$$

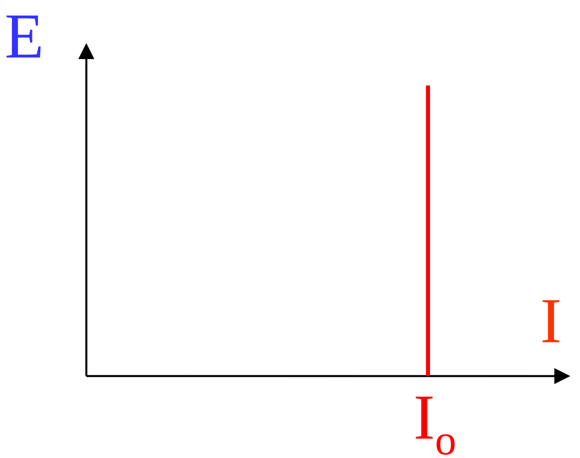
A real Voltage Source, can be modeled with an ideal voltage source (E_{oc}) and a serial resistor (internal) (R_o).

Real Current Source and Source Conversion

Power sources used may approach the ideal, but never be ideal!

Ideal Current Sources





Not Realistic!

An ideal current source tries to keep the current constant, even though the load R is infinite, which is unrealistic.

$$I_o = E_o / R$$

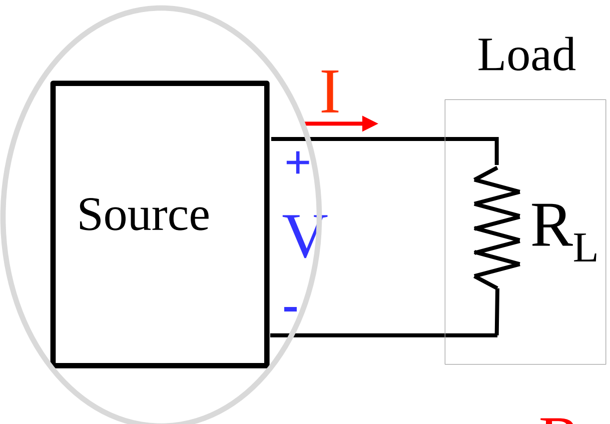
$$R \to \infty \quad I_o = 0 / \infty$$

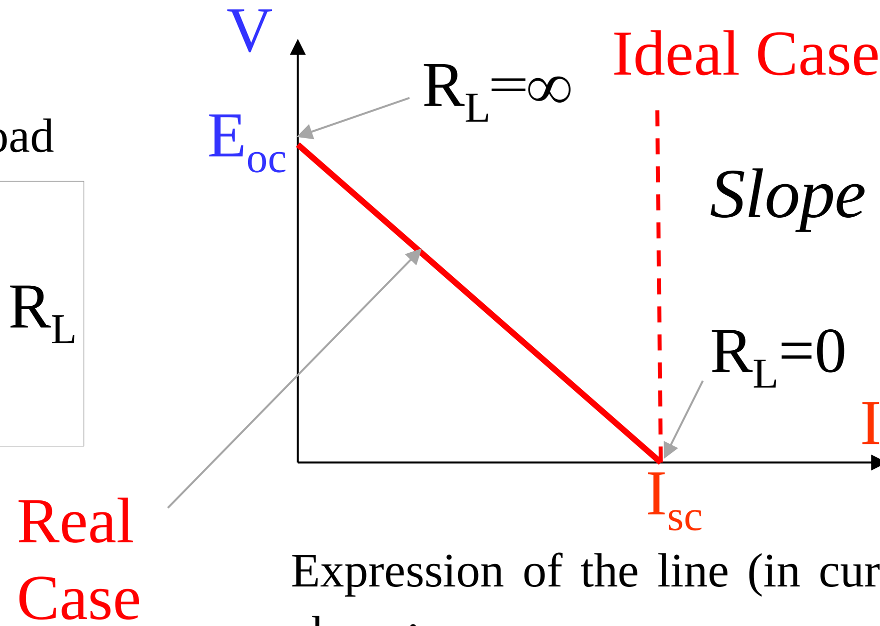
Real Current Source and Source Conversion

A real (Current) power source and its I-V curve:

E_{oc}: **O**pen **C**ircuit voltage

I_{sc}: **S**hort **C**ircuit current





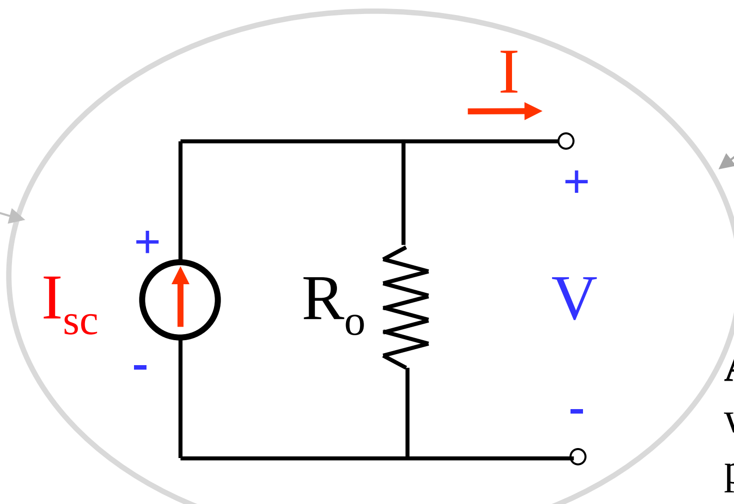
$$Slope = R_o = -\frac{E_{oc}}{I_{sc}}$$

Expression of the line (in current) in the I-V graph above:

$$I = I_{sc} - \frac{1}{R_o} V$$

Equivalent circuit for above equation

$$I = I_{sc} - \frac{1}{R_o} V \quad ..... (2)$$

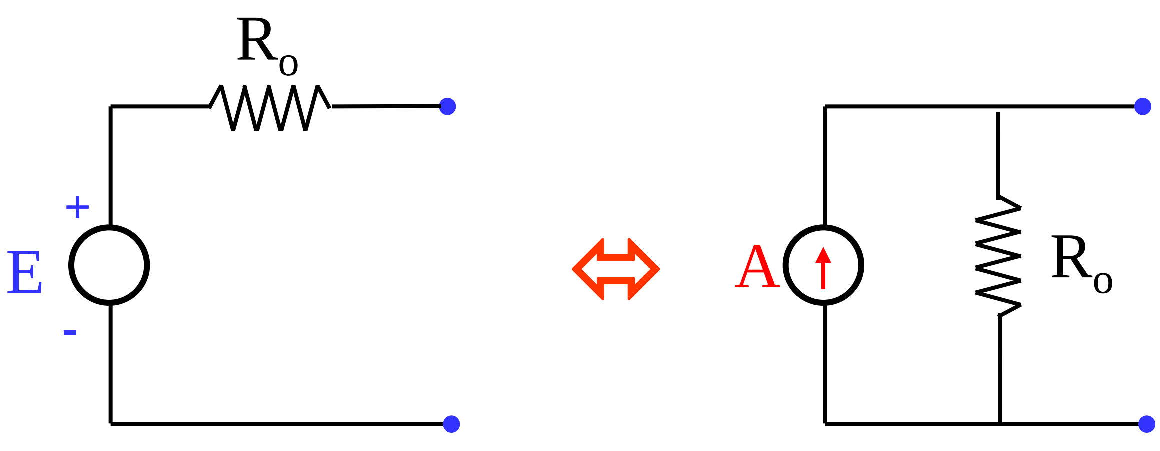


A real Current Source, can be modeled with an ideal current source (I_{sc}) and a parallel resistor (internal) (R_o).

Important Note!

Ideal sources **can not be converted** to each other. In order to be converted sources must have internal (R_o) resistance!

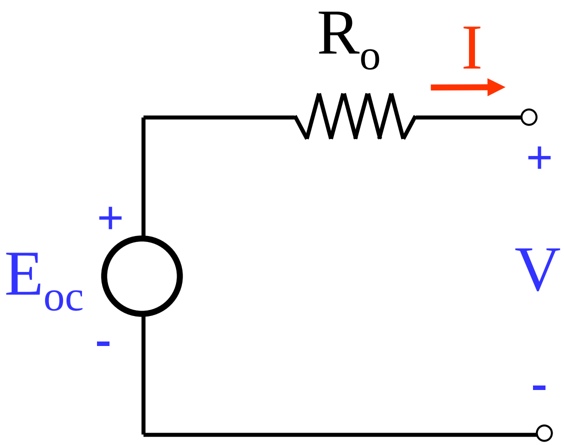
Ideal source



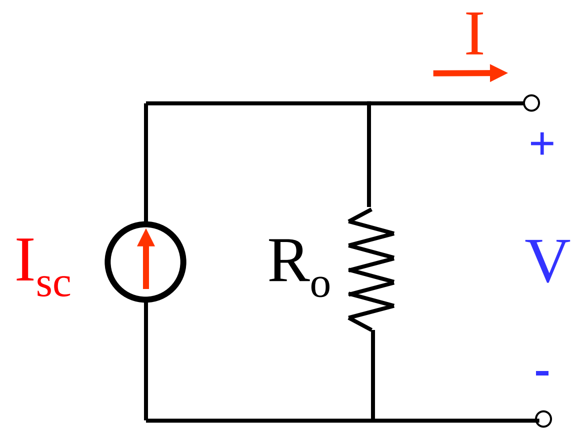
Real sources

Source Conversion: Voltage to Current

Devrelerin her ikisinin de aynı fiziksel kaynağı gösterdiğinden çıkış ucu grafiği (I ve E) özdeştir ve biri diğerini temsil etmek üzere kullanılabilir.



⇔



Voltage Source

Current Source

$$V = E_{oc} - R_o I \quad ..... (1)$$

$$I = I_{sc} - \frac{1}{R_o} V \quad ..... (2)$$

$$R_o = \frac{E_{oc}}{I_{sc}}$$

Source Conversions

To convert any Voltage Source indication to Current Source indication:

Current Source: $I = I_{sc} - \frac{1}{R_o}V = I_{sc} - G_o V$ $\qquad I = \frac{E_{oc}}{R_o} - \frac{V}{R_o}$

If: $I_{sc} = \frac{E_{oc}}{R_o}$ *and* $G_o \equiv \frac{1}{R_o}$ Then the above circuits are identical to the I-V graphs.

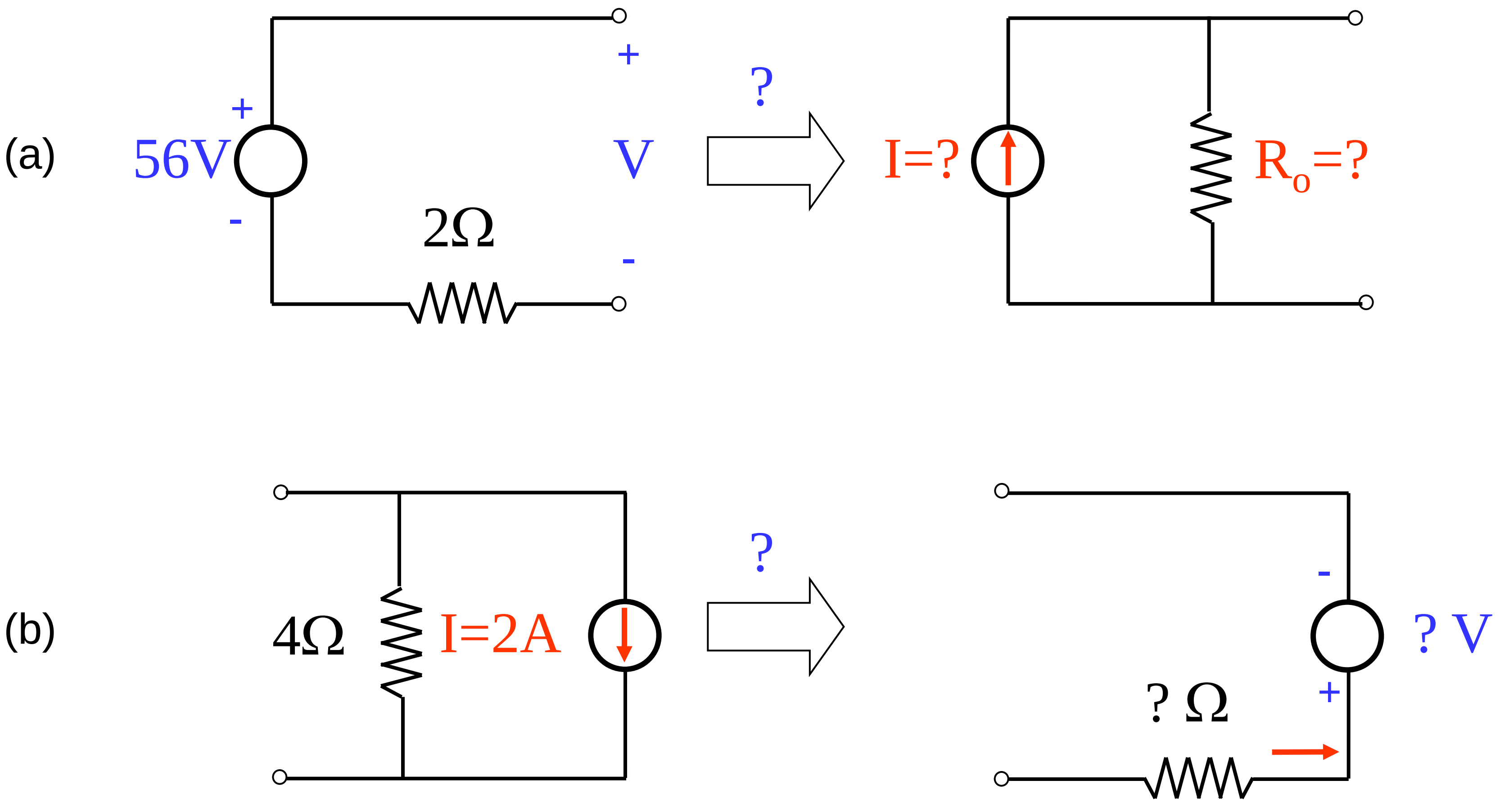
Voltage Source:

To convert any Current Source indication to Voltage Source indication :

$$V = \frac{I_{sc}}{G_o} - \frac{I}{G_o}$$

Eğer: $E_{oc} = \frac{I_{sc}}{G_o}$ *and* $R_o \equiv \frac{1}{G_o}$ Then the above circuits are identical to the I-V graphs.

Example-2.5: Convert the following voltage source to an equivalent current source (a) and the current source to an equivalent voltage source (b).



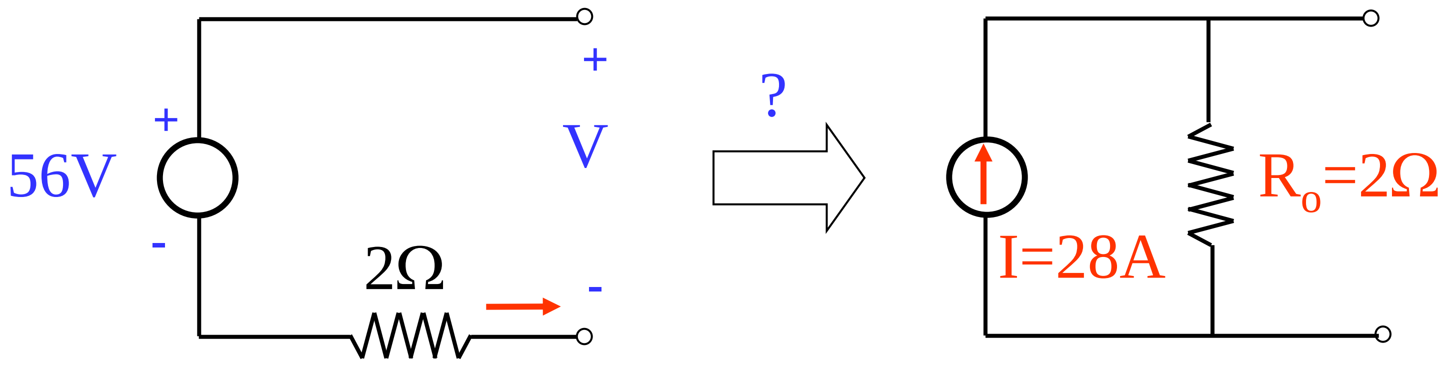
Solution: (a)

V_{oc}=56V R_o=2Ω

$$V_{oc} = R_o I_{sc} \Rightarrow (56\ V) = (2\ \Omega) I_{sc}$$

$$I_{sc} = 28\,A$$

$$G_o = \frac{1}{R_o} = \frac{1}{2\Omega} = 0,5\,mho$$



Solution: (b) $I_{sc}=2A$ $R_o=4\Omega$

$$V_{oc} = R_o I_{sc} \Rightarrow V_{oc} = (4\Omega)(2A) = 8V$$

$$G_o = \frac{1}{R_o} = \frac{1}{4\Omega} = 0.25\,mho$$