Ankara University
Engineering Faculty
Department of Engineering Physics

PEN207

Circuit Design and Analysis

Prof. Dr. Hüseyin Sarı

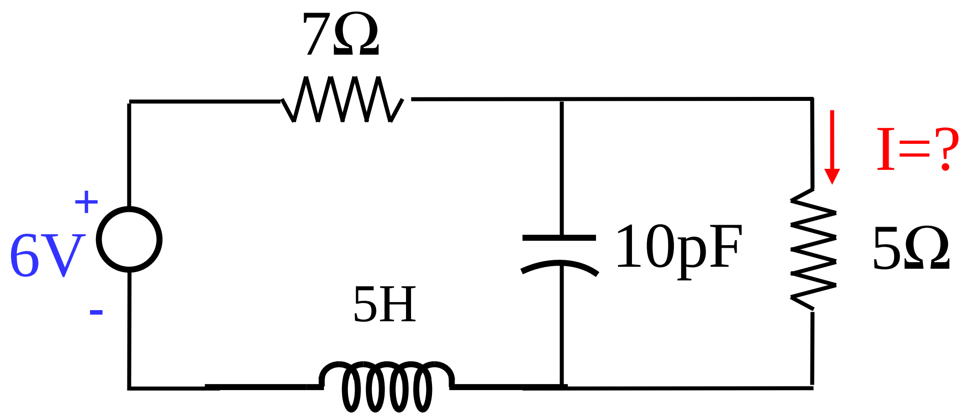
Chapter-3

Circuit Responses
(1/2)

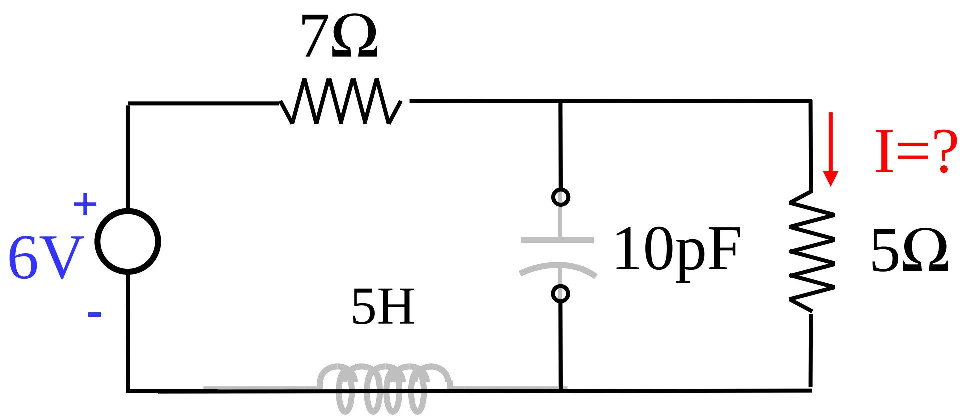
Circuit Responses
Content

- Characteristics of Circuit Response
- Natural Response
- Response of More Complex Circuits
- Forced Response
- Initial Conditions
- Exact Response

Circuit Responses



What is the current on the 5Ω?



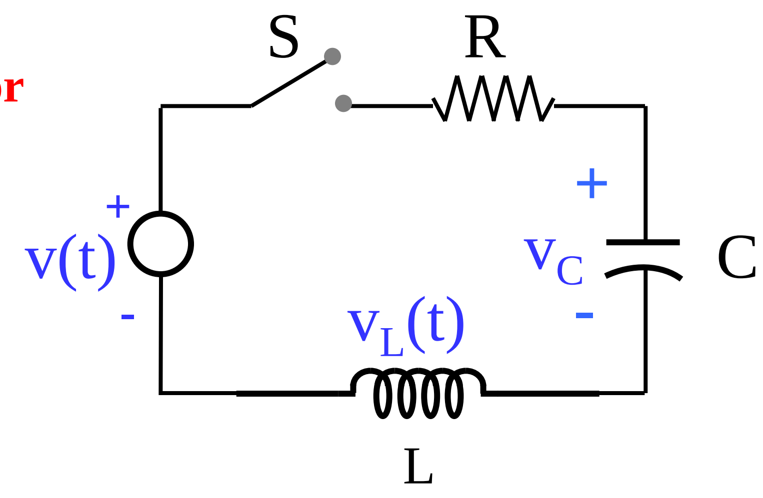
$$I = \left(\frac{6V}{12\Omega}\right) = 0.5A$$

Circuit Responses

In this chapter, the response of circuit will be examined if sudden change of current and voltage are applied to the circuits.

Current on capacitor

$$i(t) = C\frac{dv_C(t)}{dt}$$



Voltage on coil

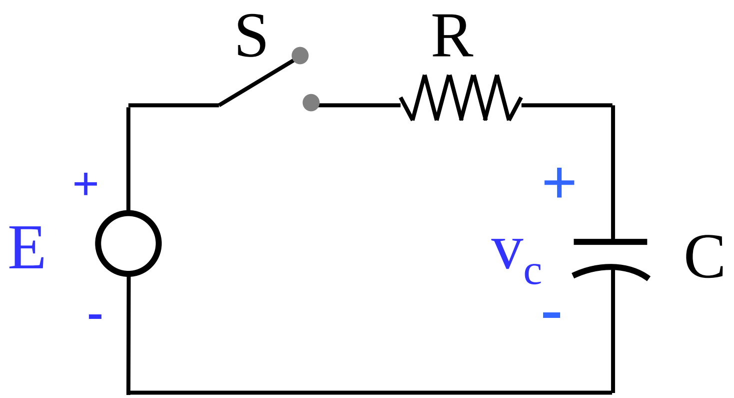
$$v_L(t) = L\frac{di(t)}{dt}$$

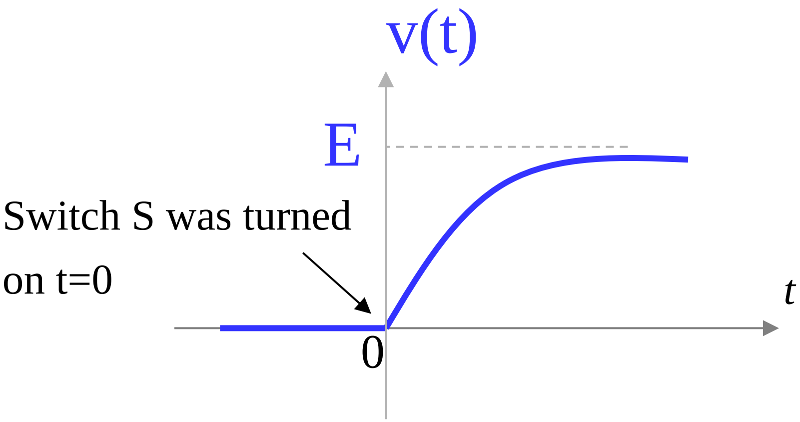
$v_C(t)=?$ $v_L(t)=?$

The current and voltage on capacitor and inductor are not proportional to the magnitude of the current and voltage, but are proportional to change on current and voltage.

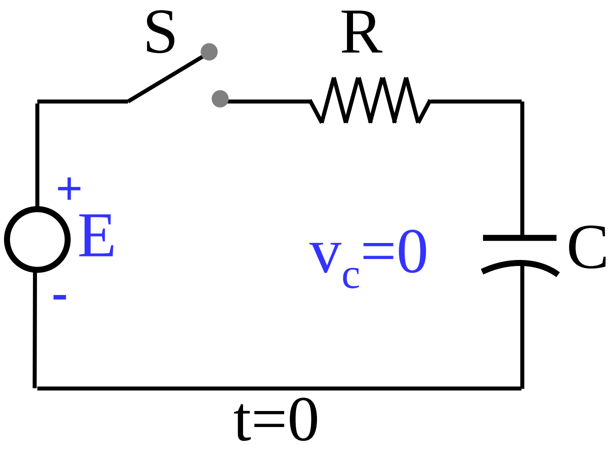
Characteristics of Circuit Response

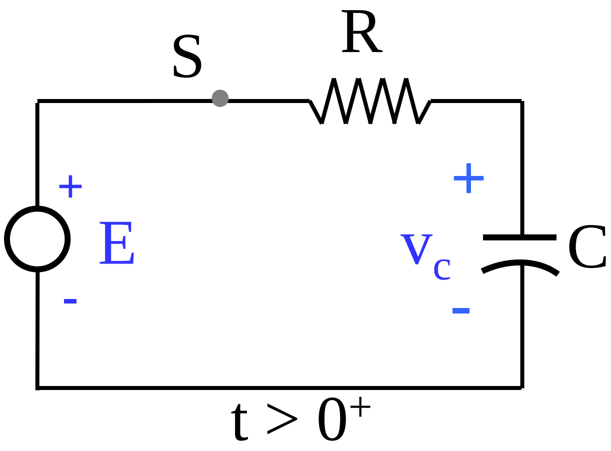
The general characteristics of the circuit response will be explained by examining the circuit consisting of resistor R and capacitor C, below. Suppose that switch S is off until t=0 and the capacitor C is empty. There is no current in the circuit before t=0. The circuit is stable.



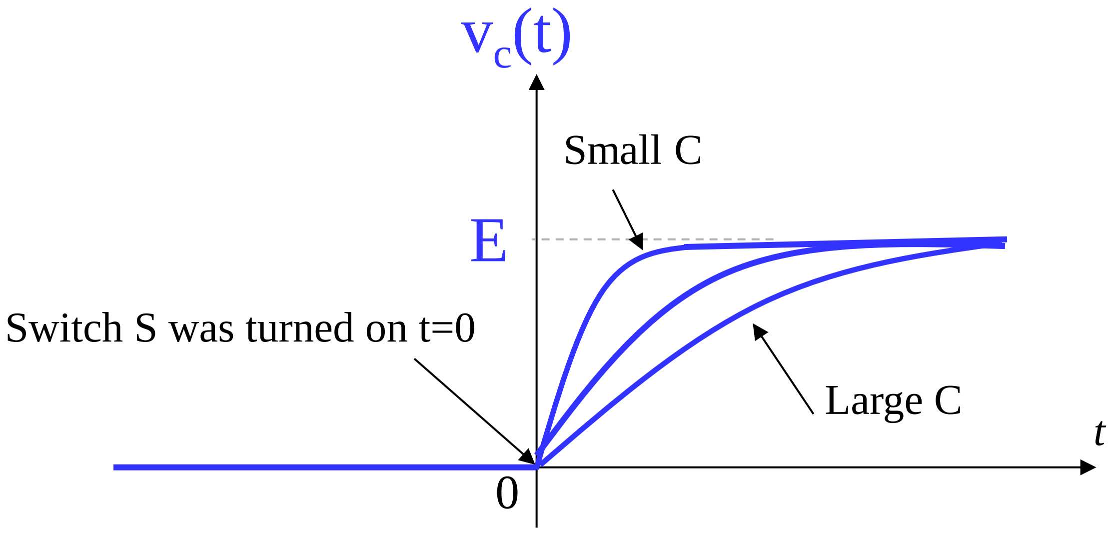


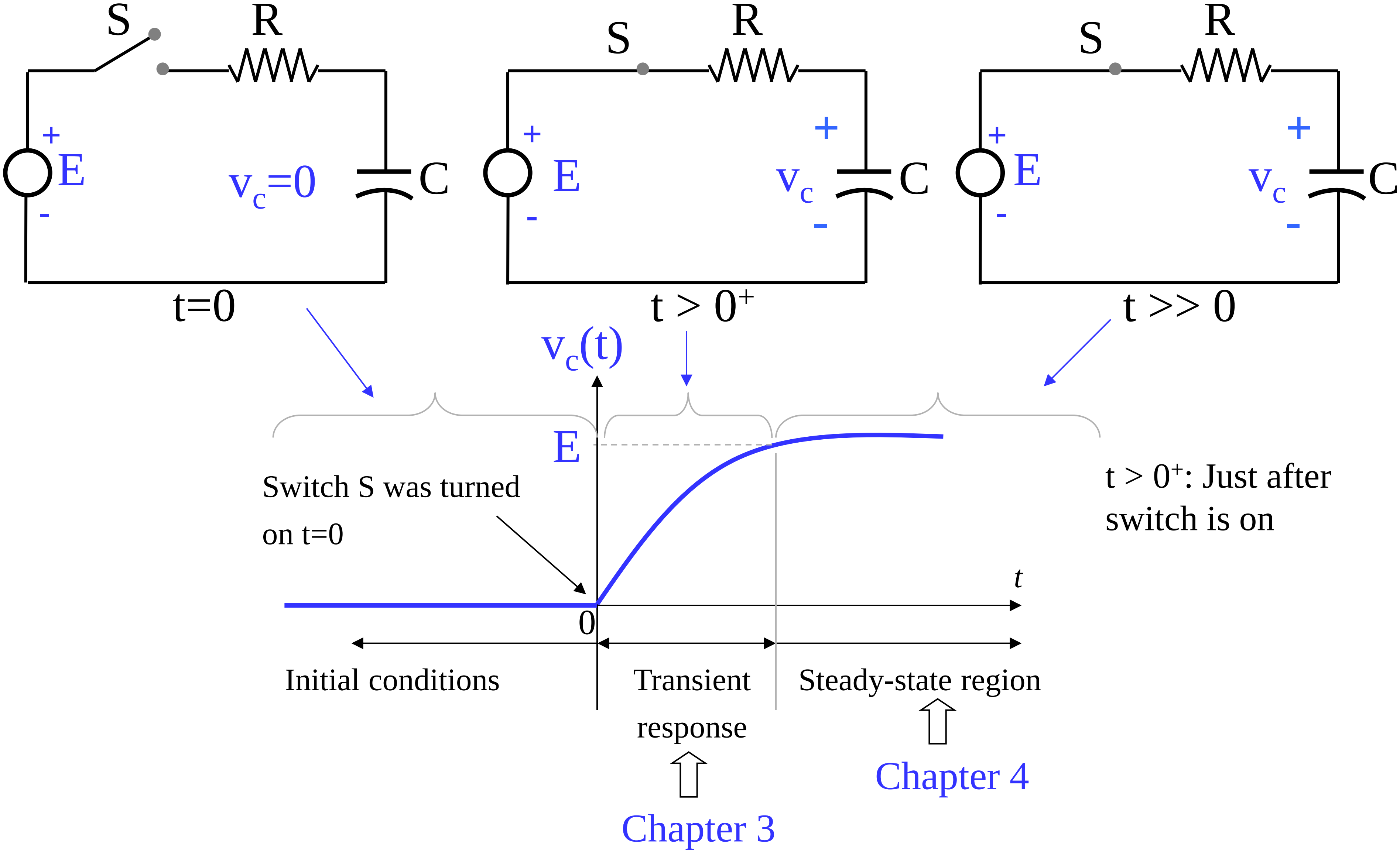
If the switch is turned on at t = 0, the conditions start to change. Charges from the voltage source start to flow through the circuit and reach the capacitor; this charge flow continues (at the beginning it is fast and then it is slows down) until the voltage (v_C) on the capacitor is equal to the source voltage (v_C=E); a new state is established and the flow of charges stops.





This transient period may be long or short depending on the circuit elements.

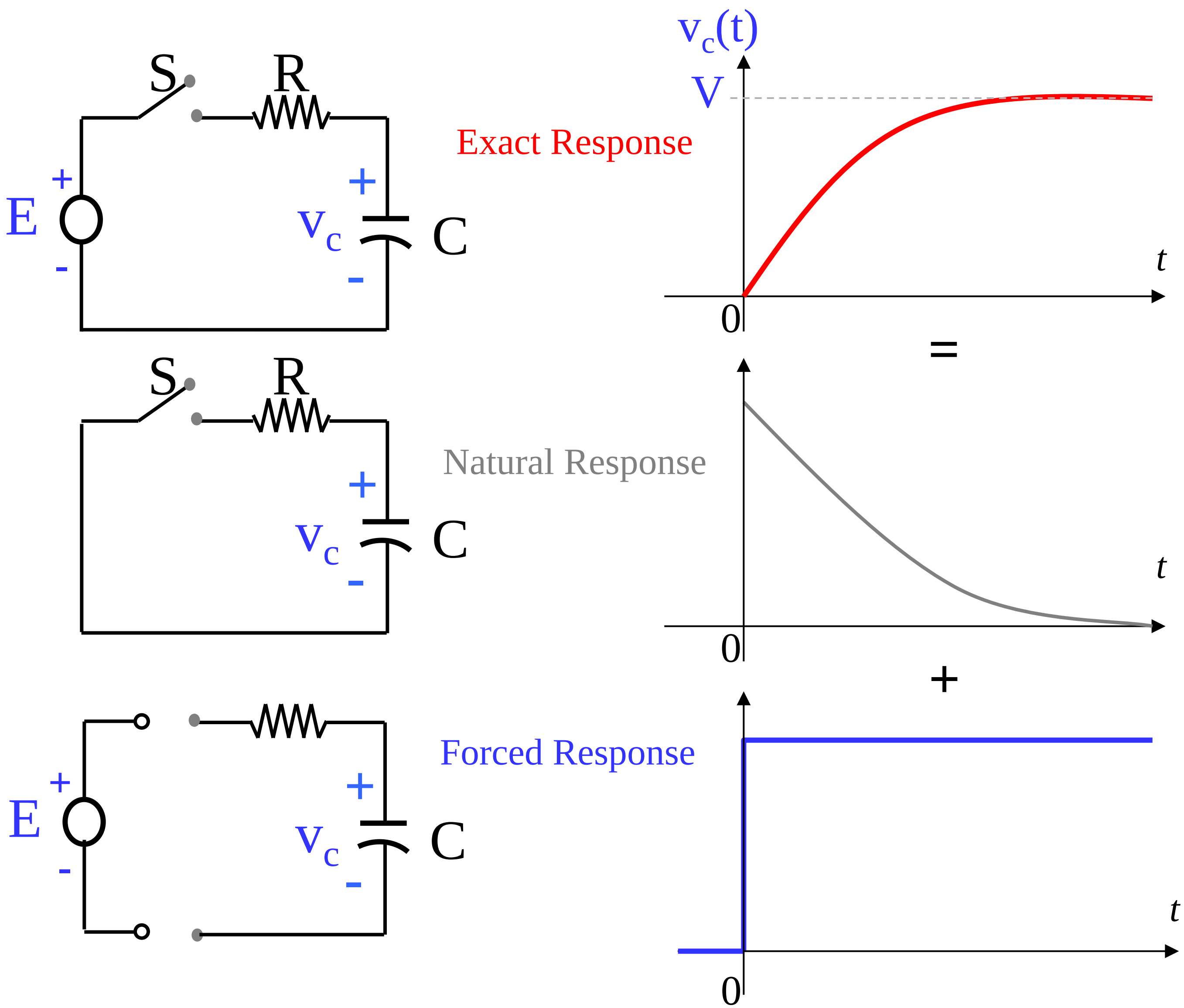




This transition period can be long or short depending on the circuit elements. This transitional period is the sum of the Natural response of the circuit elements and the Forced response generated by the power source and is the subject of this chapter.

Characteristics of Circuit Response

Full response is the sum of the Natural and Forced responses

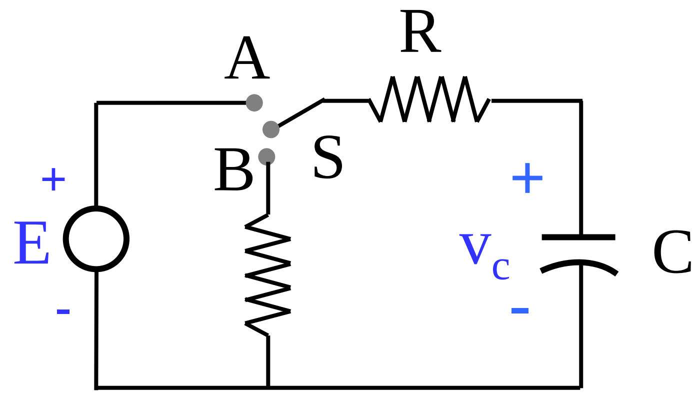


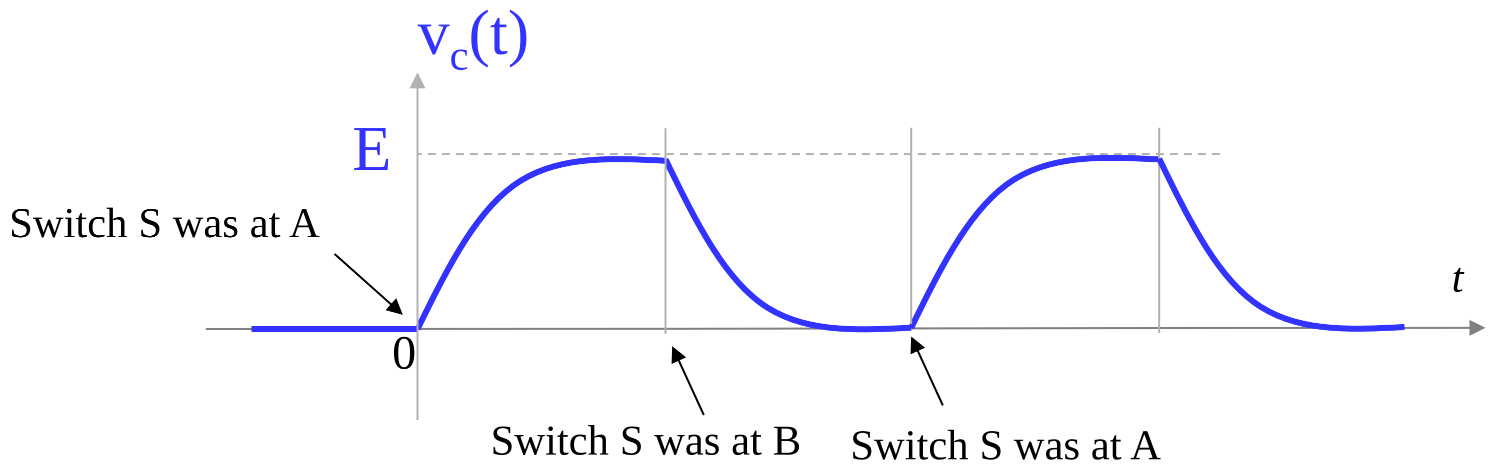
Response of the circuit observed immediately after a sudden change in the circuit (switch-off)

Natural behavior of the circuit with no power source

Voltage applied by the power source to the circuit

Characteristics of Circuit Response



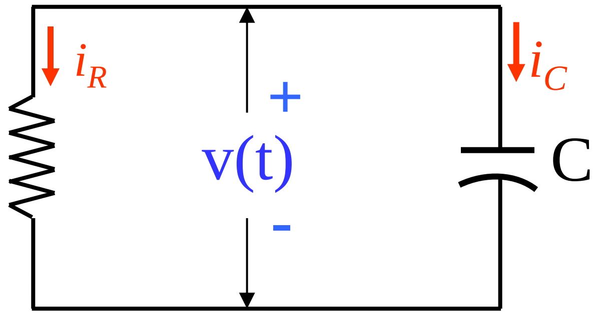


Natural Response-RC Circuit

In this section, natural response of a RC circuit, in case there is no power source, will be examined. Consider the circuit below as an example:

Current on the resistor

$$i(t) = \frac{v_R(t)}{R}$$



Current on the capacitor

$$i(t) = C\frac{dv_C(t)}{dt}$$

$$i_R + i_C = 0$$

Kirchhoff's Current Law (KCL) equation:

$$C\frac{d\mathrm{v}(t)}{dt} + \frac{1}{R}\mathrm{v}(t) = 0$$

This is first order (first derivative) homogeneous (the right side of the equation is zero) differential equation.

Since this equation contains derivatives it is called *differential equation*.

The solution must be a function whose derivative is equal to itself

$$\mathrm{v}(t) = Ke^{st}$$

If the constants s and K are known, the solution can be obtained (K can be found from initial conditions).

Remainder:

Derivative rule:

$$y(t) = Ae^{bt}$$

$$\frac{dy(t)}{dt} = Abe^{bt} = by(t)$$

$$C\frac{d}{dt}(Ke^{st}) + \frac{1}{R}Ke^{st} = 0 \quad \Rightarrow \quad Ke^{st}(sC + \frac{1}{R}) = 0$$

$$sC + \frac{1}{R} = 0 \quad \Rightarrow \quad s = -\frac{1}{RC} \quad \Rightarrow \quad \mathrm{v}(t) = Ke^{-t/RC}$$

Natural Response-RC Circuit

$$v(t) = Ke^{st} \qquad s = -\frac{1}{RC}$$

$$s = -\frac{1}{RC}$$

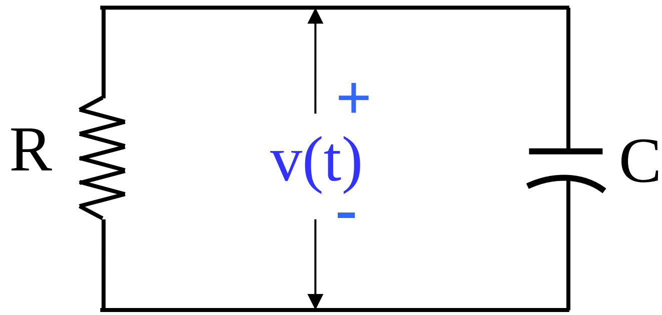
s is in 1/time dimension and what physically it means will be discussed in detail in the following sections.

$$v(t) = Ke^{-t/RC}$$

The value of the constant K is determined by using the initial value of v(t). In this case, since v(t=0)=V_o (v(t)=V_o at t=0), K=V_o is found.
(If the capacitor were initially empty then v(t=0)=0, K=0)

In this case solution (Natural response of the circuit):

$$v(t) = V_o e^{-t/RC}$$

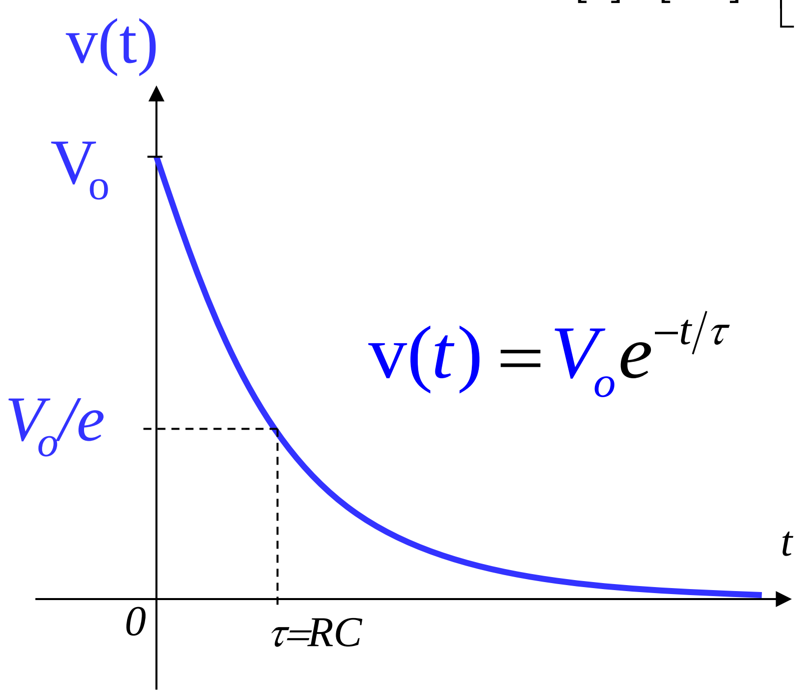


$$v(t) = Ke^{-t/RC}$$

$$\tau \equiv RC$$

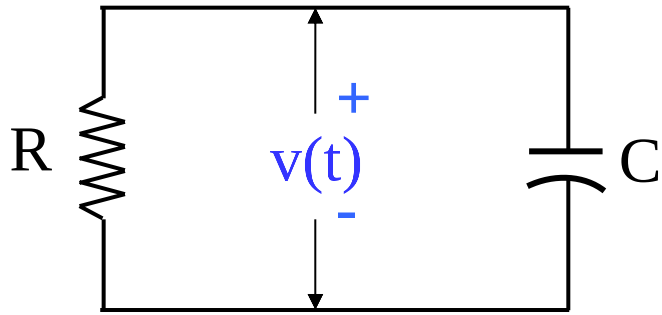
τ is in time dimension

$$[\tau] = [RC] = \left[\frac{V}{I}\right]\left[\frac{Q}{V}\right] = \left[\frac{V}{Q/T}\right]\left[\frac{Q}{V}\right] = [T]$$



$$v(t) = V_o e^{-t/\tau}$$

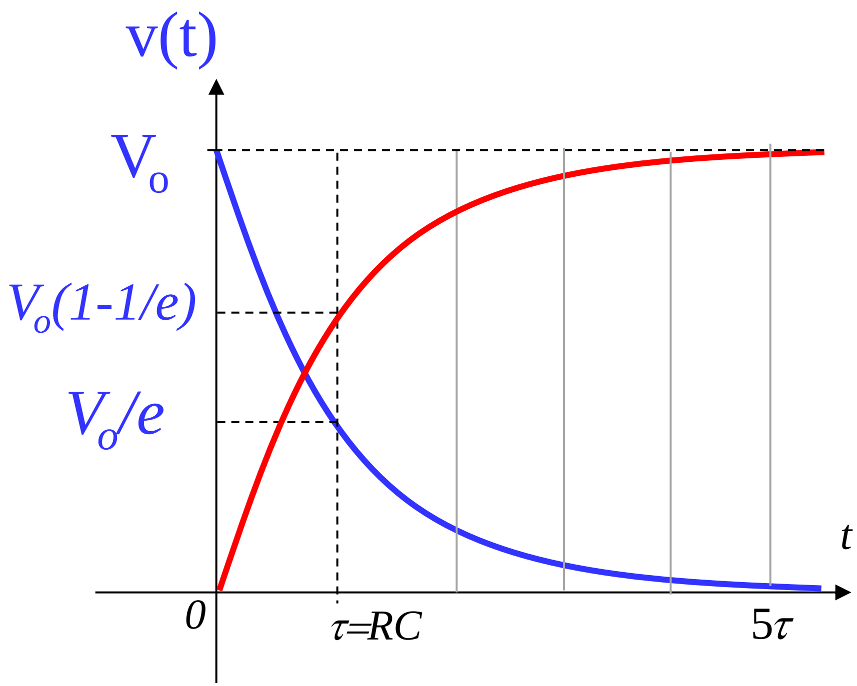
$\tau = RC$; the time constant it takes for the initial voltage (V_o) to fall to the V_o/e value



$$v(t) = Ke^{-t/RC}$$

$$v(t) = V_o e^{-t/\tau}$$

$$\tau \equiv RC$$

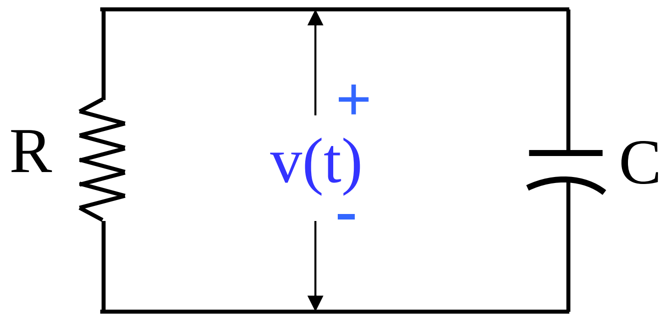


τ= RC; the time constant it takes for the initial voltage (V_o) to fall to the V_o/e value

Charging or discharging phase of a capacitor has ended after 5 τ

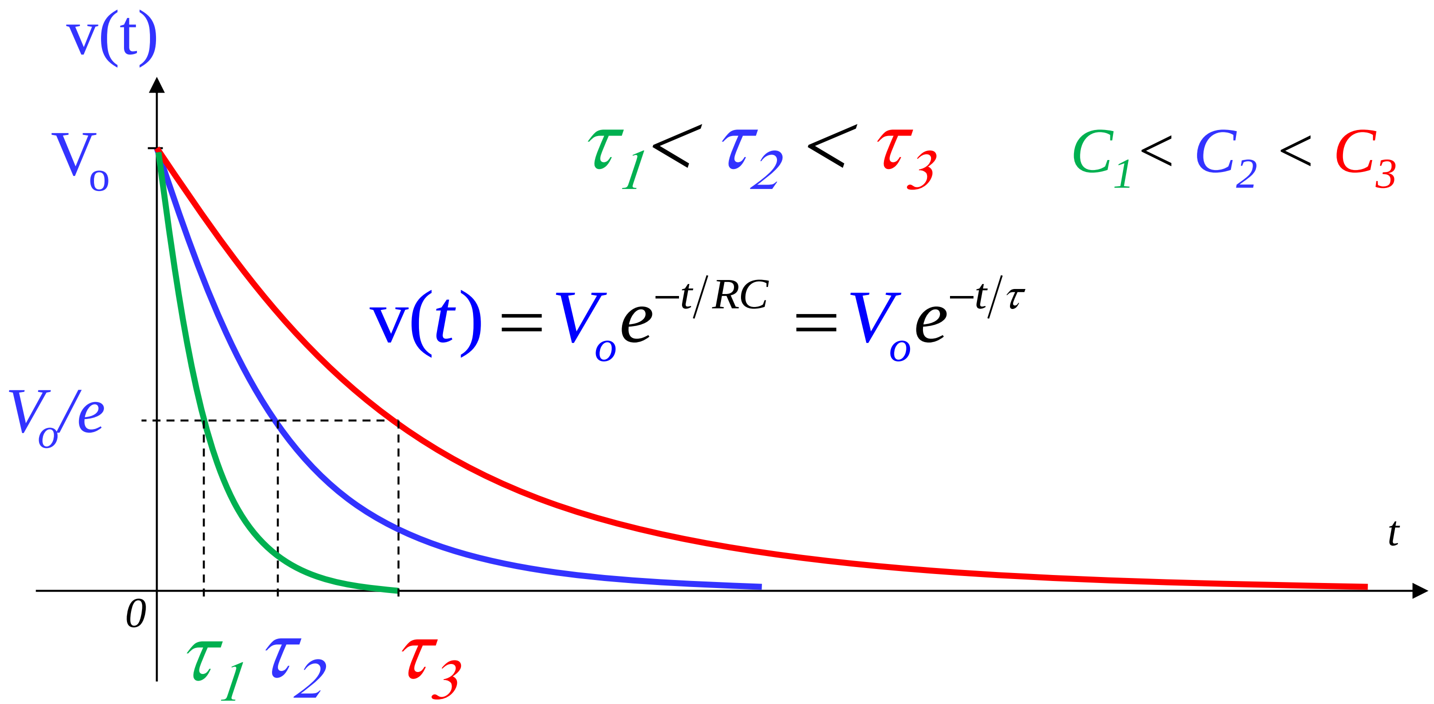
Charging: $v(t) = V_o(1-e^{-t/\tau}) \Rightarrow v(t=5\tau) = V_o(1-e^{-5\tau/\tau}) = V_o(1-e^{-5}) = V_o(1-0.007) \cong (0.993)V_o$

Discharging: $v(t) = V_o e^{-t/\tau} \Rightarrow v(t=5\tau) = V_o e^{-5\tau/\tau} = V_o e^{-5} = V_o(0.007) \cong (0.003)V_o$



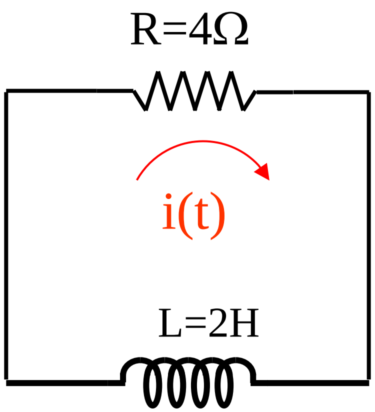
$$\tau = RC$$

$$v(t) = Ke^{-t/RC}$$

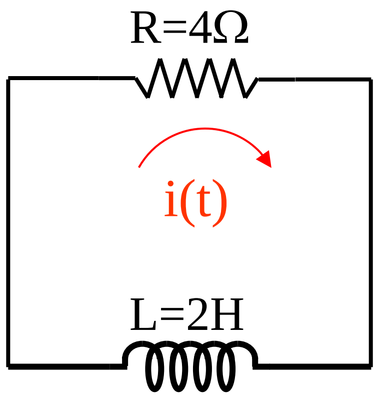


Example-3.1: In the RL circuit below:

(a) find the natural response i(t)
(b) If i(0)=5A find the numerical value of i(t).



Solution:(a) Kirchhoff's Voltage Law(KVL)



$$V_R + V_L = 0$$

$$L\frac{di(t)}{dt} + Ri(t) = 0$$

Solution of above differential equation: $i(t) = Ke^{st}$

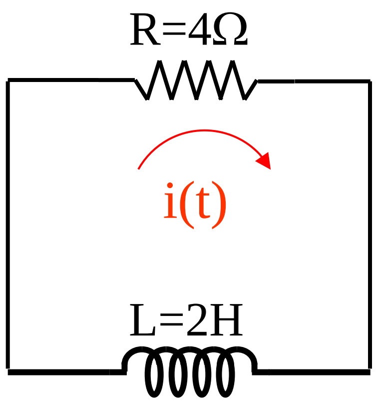
$$L\frac{d}{dt}(Ke^{st}) + R(Ke^{st}) = 0 \Rightarrow Ke^{st}(sL+R) = 0 \Rightarrow sL + R = 0 \Rightarrow s = -\frac{R}{L}$$

Solution: $i(t) = Ke^{-Rt/L}$

(b) When the values given in the problem are used (i(t=0)=5A)

$$i(t=0) = 5A = Ke^{-(4/2)0} = K.1 \Rightarrow K = 5\ A$$

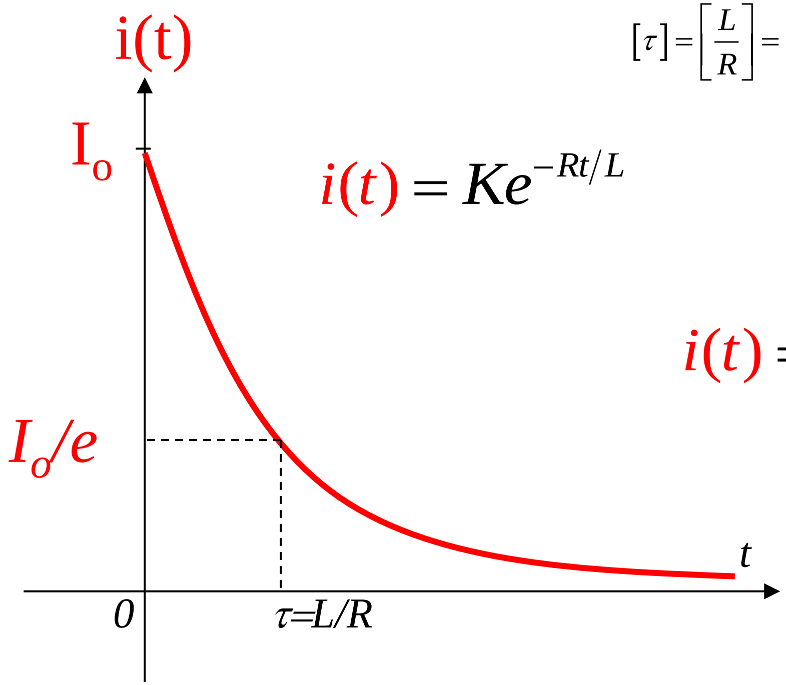
$$i(t) = 5e^{-2t}\ amper$$



$$L\frac{di(t)}{dt} + Ri(t) = 0$$

τ is in time dimension

$$[\tau] = \left[\frac{L}{R}\right] = \left[\frac{L}{V/I}\right] = \left[\frac{V/(I/T)}{V/I}\right] = \left[\frac{1}{V/I}\right]\left[\frac{V}{I/T}\right] = [T]$$

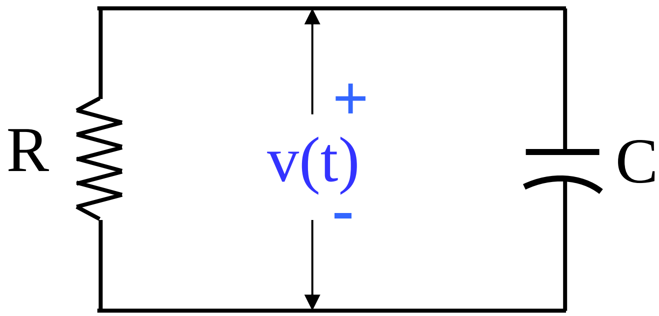


$$i(t) = Ke^{-Rt/L}$$

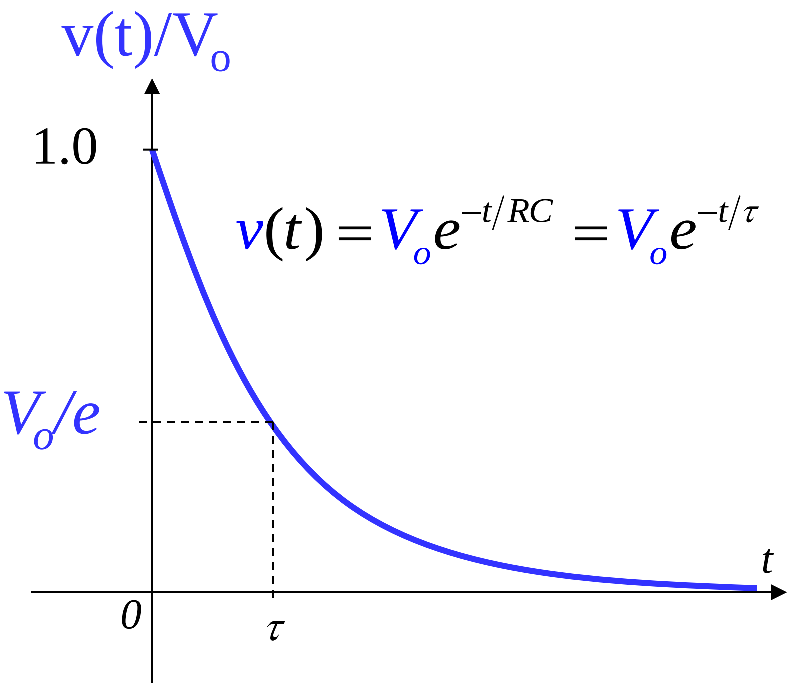
$$i(t) = Ke^{-Rt/L} = Ke^{-t/\tau}$$

τ= L/R;
the time constant it takes for the initial current (I_o) to fall to the I_o/e value

RC Circuit



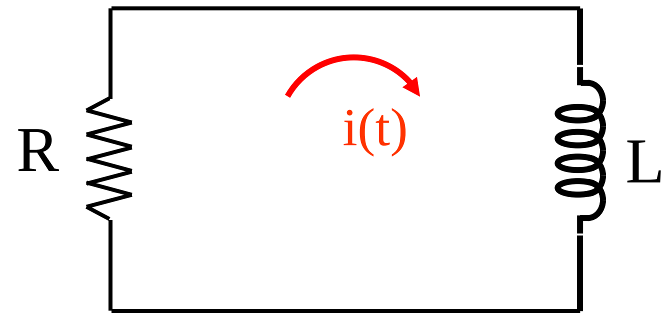
$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)=0$$



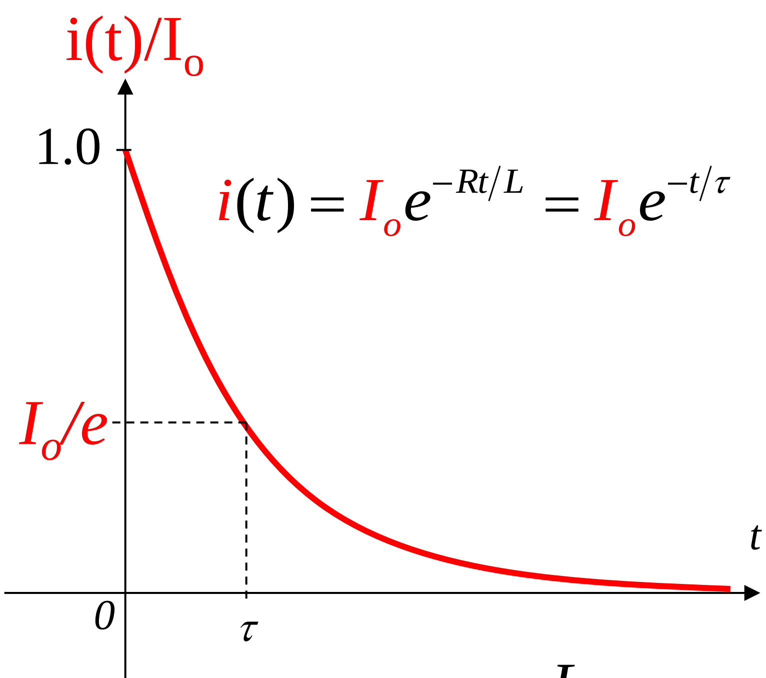
$$v(t)=V_o e^{-t/RC}=V_o e^{-t/\tau}$$

Time Constant $\tau \equiv RC$

RL Circuit



$$L\frac{di(t)}{dt}+Ri(t)=0$$



$$i(t)=I_o e^{-Rt/L}=I_o e^{-t/\tau}$$

Time Constant $\tau \equiv \frac{L}{R}$

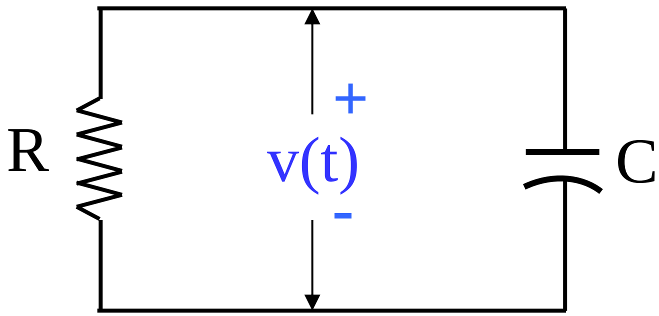
$$i(t)=Ke^{-\alpha t}$$

$$\alpha \equiv \frac{1}{\tau}$$

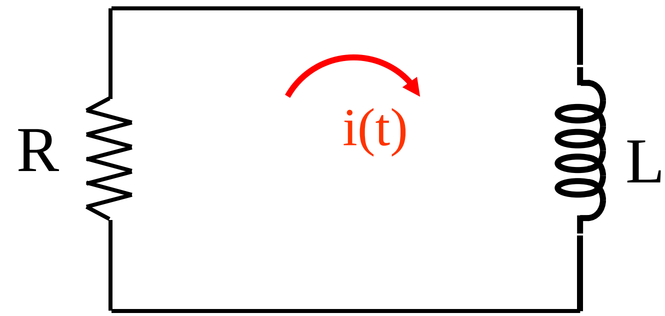
τ Time constant:

A scale of the length of time required to disaapear the natural response

RC Circuit



RL Circuit



KCL $$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)=0$$

$$v(t)=V_o e^{-t/RC}$$

$$i(t)=v(t)/R=\frac{V_o}{R}e^{-t/RC}=I_o e^{-t/RC}$$

KVL $$Ri(t)+\frac{1}{C}\int i(t)dt=0$$

$$R\frac{di(t)}{dt}+\frac{1}{C}i(t)=0$$

$$i(t)=I_o e^{-t/RC}$$

$$v(t)=i(t)R=I_o Re^{-t/RC}=V_o e^{-t/RC}$$

KVL $$L\frac{di(t)}{dt}+Ri(t)=0$$

$$i(t)=I_o e^{-Rt/L}$$

$$v(t)=Ri(t)=RI_o e^{-Rt/L}=V_o e^{-Rt/L}$$

KCL $$\frac{1}{R}v(t)+\frac{1}{L}\int dv(t)dt=0$$

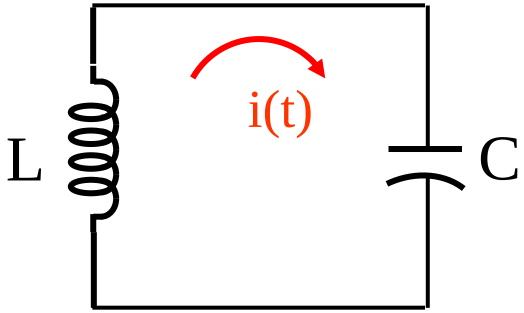
$$\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=0$$

$$v(t)=V_o e^{-Rt/L}$$

$$i(t)=\frac{v(t)}{R}=\frac{V_o}{R}e^{-Rt/L}=I_o e^{-Rt/L}$$

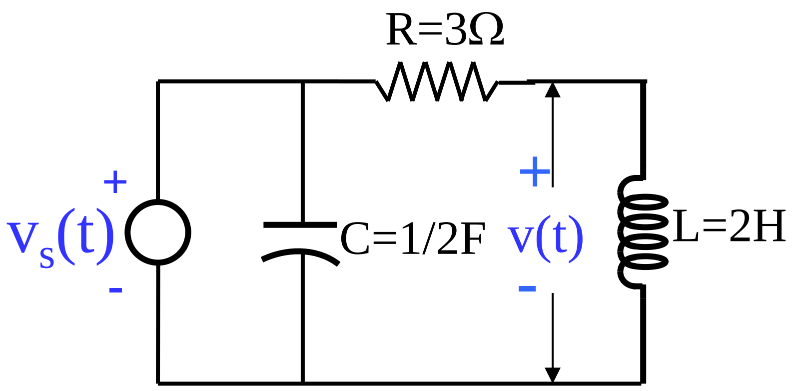
Homework: Find the natural response of component of current i(t) of the LC circuit below

LC Circuit

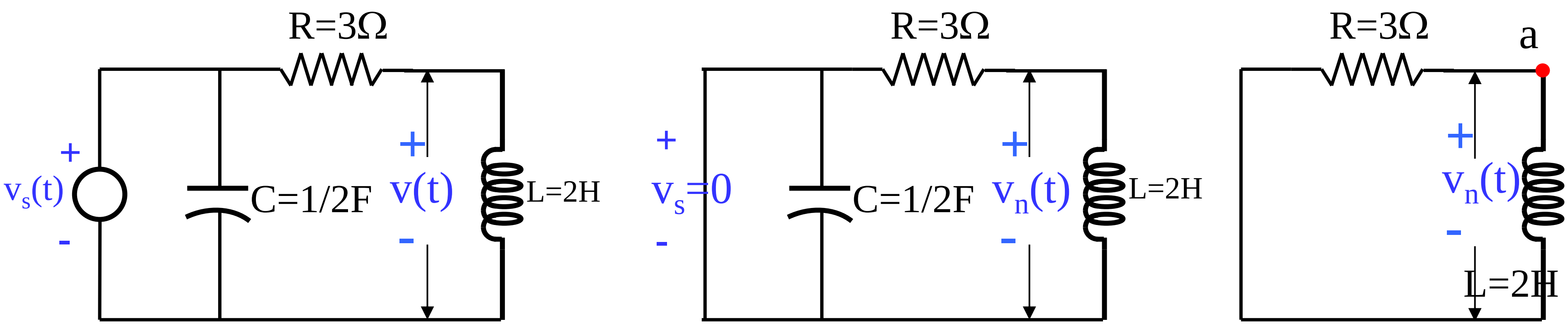


No energy consumption (no resistance in the circuit!)

Example-3.2: Find the natural component of the v(t) response of the circuit below.



Solution-3.2: The first step is to remove the effect of the source, since the natural response component is asked. If the voltage source is short-circuited, the effect of the source is removed.



The voltage at the terminals of the capacitor is forced to be zero due to the short circuit.

KCL: $\frac{1}{3}\mathrm{v_n}(t)+\frac{1}{2}\int \mathrm{v_n}(t)dt=0$ $\quad \frac{1}{3}\frac{d\mathrm{v_n}(t)}{dt}+\frac{1}{2}\mathrm{v_n}(t)=0$ First order differential equation

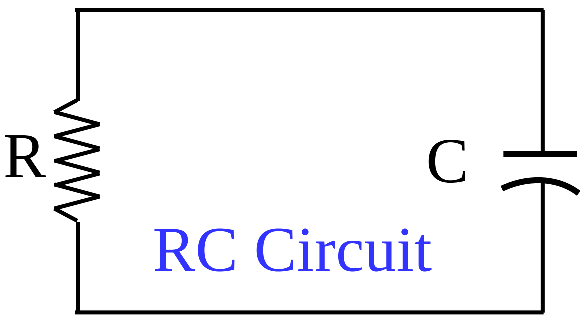
Solution: $\mathrm{v_n}(t)=V_o e^{st}$

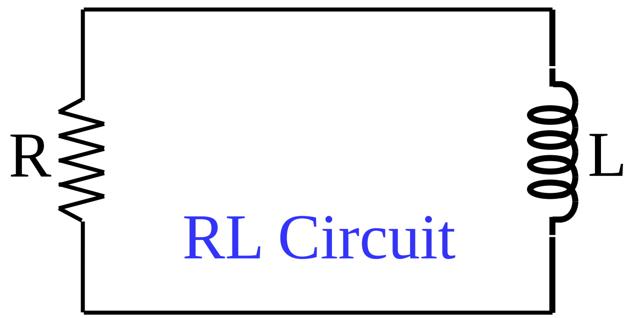
$$\frac{1}{3}\frac{d}{dt}(V_o e^{st})+\frac{1}{2}V_o e^{st}=0 \Rightarrow V_o e^{st}(\frac{s}{3}+\frac{1}{2})=0 \Rightarrow \frac{s}{3}+\frac{1}{2}=0 \Rightarrow s=-1.5$$

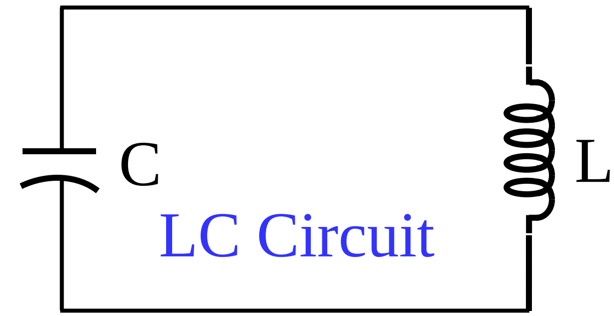
$$\boxed{\mathrm{v_n}(t)=V_o e^{-(1.5)t}}$$

Natural Response of More Complex Circuits

In the previous section, the circuit contained only one circuit element (capacitor or inductance) that stored the energy.



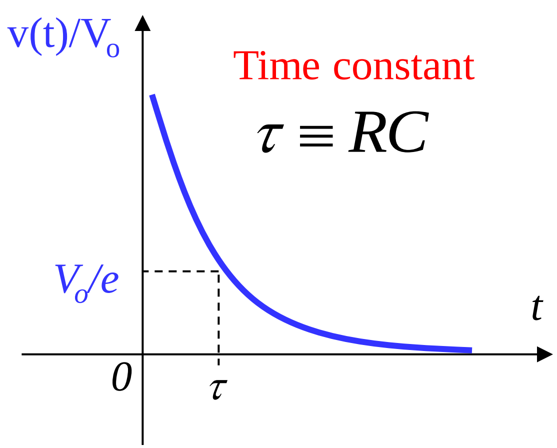


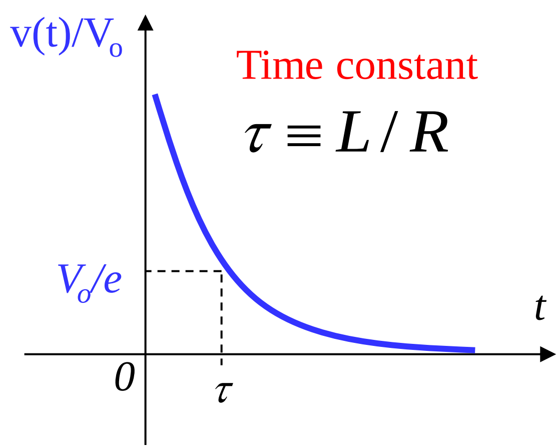


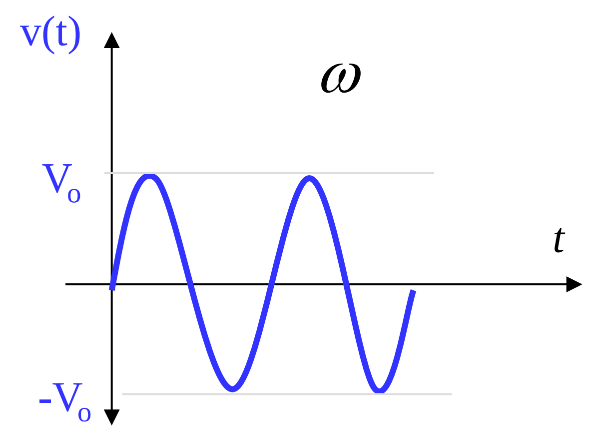
Energy loss!

Energy loss!

No energy loss!

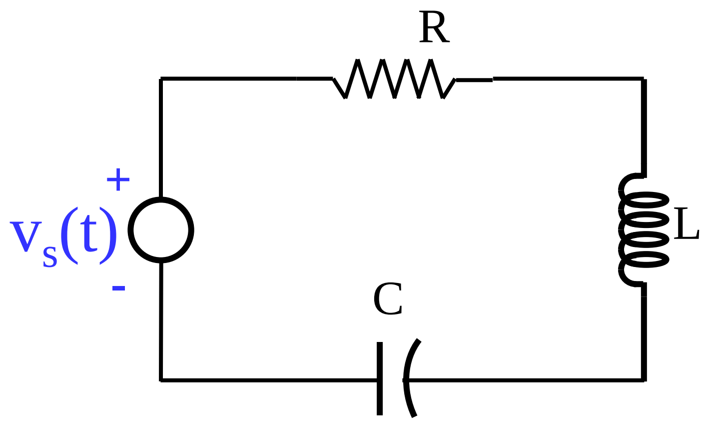






Natural Response of More Complex Circuits

In the previous section, the circuit contained single circuit element (capacitor or inductor) that store energy. In this section, the natural responses of the circuits containing more than one circuit elements storing energy will be examined.



Kirchhof's Voltage Law (KVL):

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = v_s(t)$$

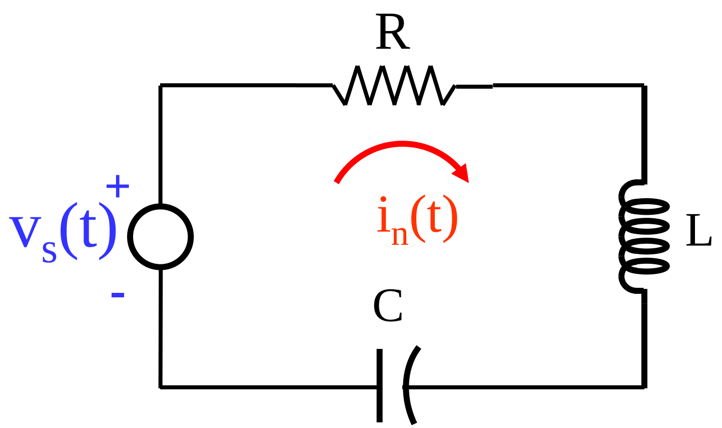
Take derivative of both side to get rid of the integral form

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{v_s(t)}{dt}$$

This is a second order (second derivation) non-homogeneous (right side of the equation is non-zero) differential equation.

Natural reponse of the circuit, $i_n(t)$ (in case of $v_s(t)=0$)

$$L\frac{d^2 i_n(t)}{dt^2} + R\frac{di_n(t)}{dt} + \frac{1}{C}i_n(t) = 0$$



Natural response of the circuit $i_n(t)$

$$L\frac{d^2 i_n(t)}{dt^2} + R\frac{d i_n(t)}{dt} + \frac{1}{C} i_n(t) = 0$$

Solution: $i_n(t) = Ke^{st} \Rightarrow s^2 LKe^{st} + sRKe^{st} + \frac{1}{C}Ke^{st} = 0$

$$Ke^{st}(s^2 L + sR + \frac{1}{C}) = 0 \quad s^2 L + sR + \frac{1}{C} = 0$$

Since it is a quadratic, in general there are two roots; s_1 and s_2

$$i_n(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

K_1 and K_2 coefficients can be found from initial conditions.

Short remainder about differential equations and their solution:

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=0$$

Solution:

$$y(t)=Ae^{st}$$

$$aAs^2e^{st}+bAse^{st}+ce^{st}=Ae^{st}(as^2+bs+c)=0$$

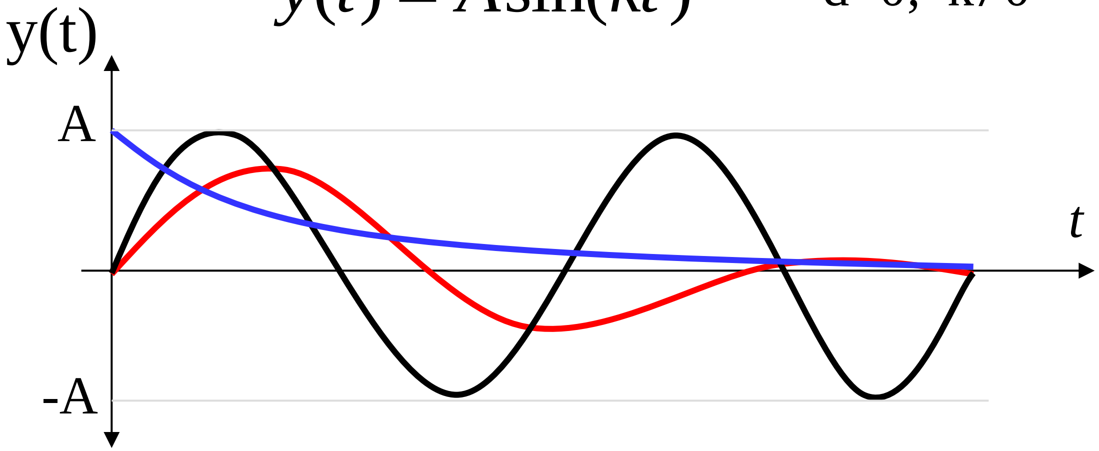
$$as^2+bs+c=0 \quad \Rightarrow \quad s_{1,2}=d\pm jk$$

In general roots are complex number

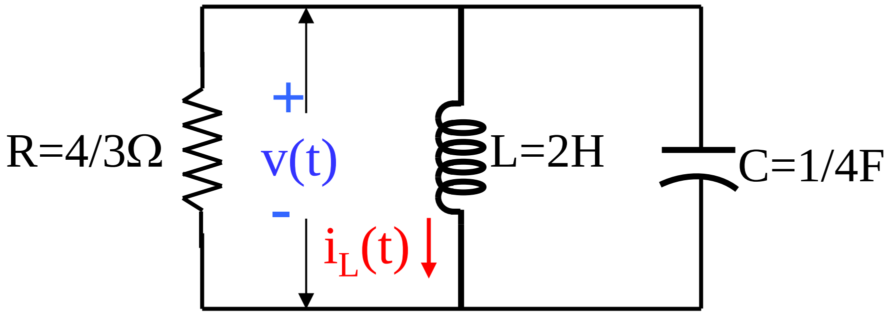
$y(t)=Ae^{-dt}$ $d\neq0$; $k=0$ (Real)

$y(t)=Ae^{-dt}\sin(kt)$ $d\neq0$; $k\neq0$ (Complex)

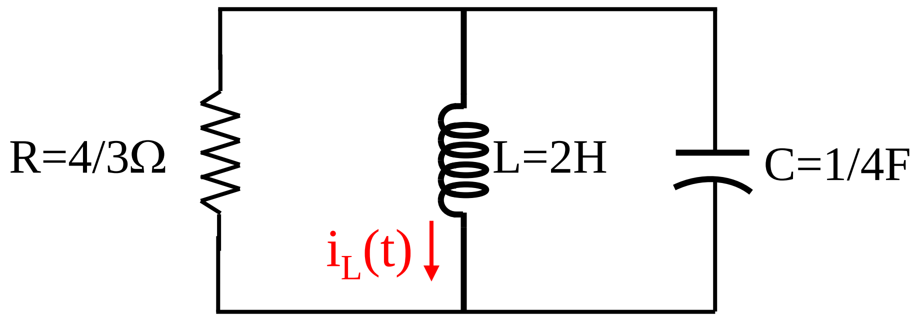
$y(t)=A\sin(kt)$ $d=0$; $k\neq0$ (Imaginary)



Example-3.3: The following circuit is triggered by a current source disconnected at t=0. At the moment the current source is disconnected, the capacitance voltage is 4V and inductance current (i_L) is 1A. Find the voltage v(t) for t> 0.



Solution: The response is the natural response since the source is disconnected at t=0. First the solution of the natural response of the circuit will be found and then the coefficients will be determined using the initial conditions.



Natural response of the circuit ($v_s(t)=0$):

Kirchhoff's Current Law (KCL): $\frac{3}{4}v(t)+\frac{1}{4}\frac{dv(t)}{dt}+\frac{1}{2}\int v(t)dt=0$

$i_R+i_C+i_L=0$

$\frac{1}{R}v(t)+C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)dt=0$

$$ax^2+bx+c=0$$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solution: $v(t)=Ke^{st}$ ⇨ $\frac{d^2v(t)}{dt^2}+3\frac{dv(t)}{dt}+2v(t)=0$

$$Ke^{st}(s^2+3s+2)=0 \qquad s^2+3s+2=0$$

⇨ $s_{1,2}=\frac{-3\pm\sqrt{3^2-4.1.2}}{2.1}=\frac{-3\pm1}{2}$

$s_1=-2 \quad s_2=-1$

Roots are real

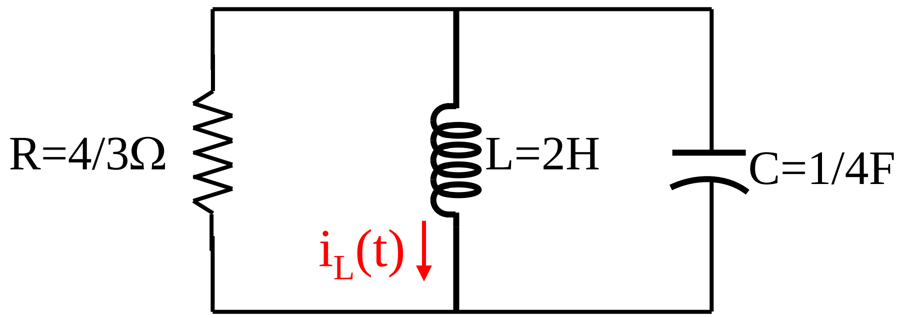
$$s_1=-2; s_2=-1$$

$$v(t)=K_1e^{-2t}+K_2e^{-t}$$

K_1 and K_2 coefficients can be found from the initial conditions (t=0).

Let's use initial conditions (t=0) to find K_1 and K_2 coefficients:

$$v(t) = K_1 e^{-2t} + K_2 e^{-t}$$



Since there are two unknowns, we must obtain two equations.

At t=0; v(0)=4V

$$v(t=0) = K_1 + K_2$$

Initial conditions:
t=0 current the capacitance voltage is 4V and inductance current (i_L) is 1A.

At t=0; derivative

$$\frac{dv(t=0)}{dt} = -2K_1 - K_2$$

What is the value of dv(t=0)/dt ?

If the v(t=0) and dv(t=0)/dt values are known at t=0 then K_1 and K_2 coefficients can be found.

Initial Conditions:

1- Voltage on capacitance is (v_c)= 4V; when the power source is disconnected

2- Current on the inductor (i_L)=1 A

$$\frac{3}{4}v(t)+\frac{1}{4}\frac{dv(t)}{dt}+\frac{1}{2}\int v(t)dt=0$$

KCL equation at t=0

$$i_R(t=0)+i_C(t=0)+i_L(t=0)=0$$

$$\frac{3}{4}v(t=0)+\frac{1}{4}\frac{dv(t=0)}{dt}+i_L(t=0)=0$$

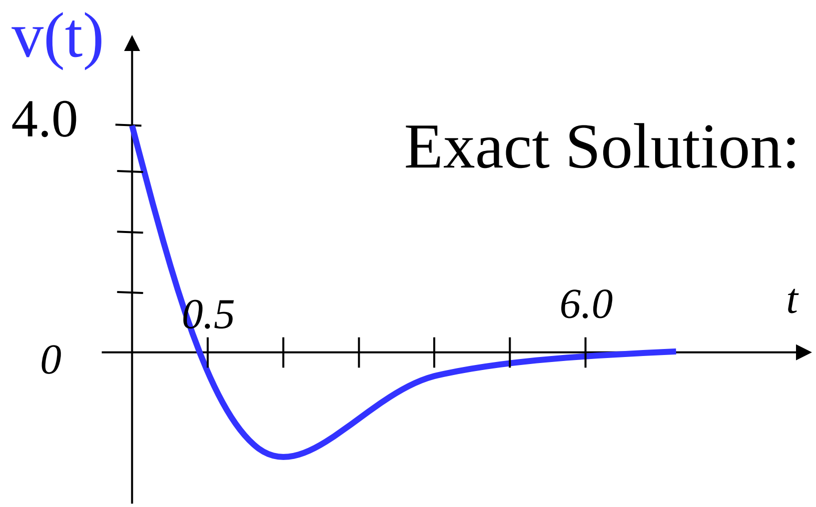
$$i_L(t=0)=1A$$

$$\frac{3}{4}4+\frac{1}{4}\frac{dv(t=0)}{dt}+1=0 \Rightarrow \frac{dv(t=0)}{dt}=-16$$

$$v(t=0)=K_1+K_2=4$$

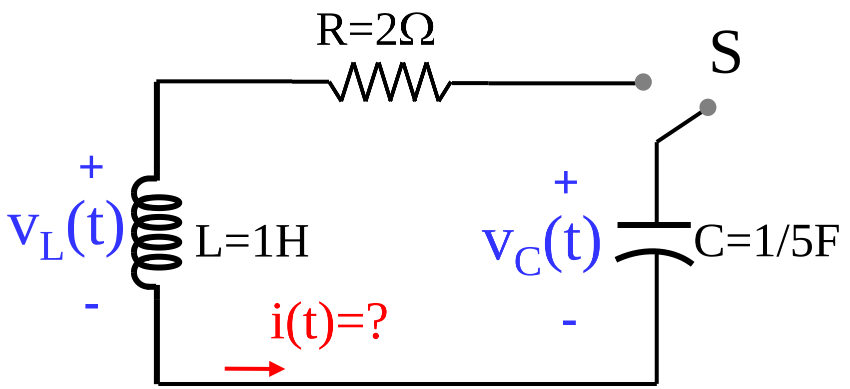
$$\frac{dv(t=0)}{dt}=-2K_1-K_2=-16$$

$$\Rightarrow \begin{aligned} K_1+K_2&=4 \\ 2K_1+K_2&=16 \\ K_1=12;\ K_2&=-8 \end{aligned}$$

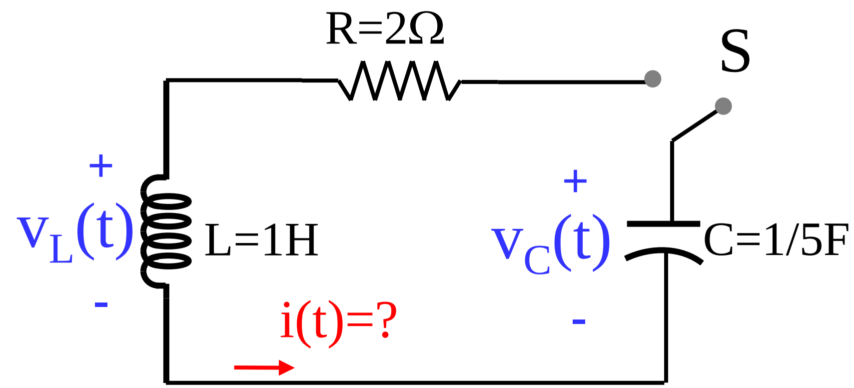


Exact Solution: $v(t)=12e^{-2t}-8e^{-t}$

Example-3.4: The 1/5F capacitor in the following circuit is charged with a second circuit (not shown) before time t=0. Since the capacitor is charged enough before t<0, the voltage on it will be v_c=10V. Find the current **i(t)** in the circuit for t>0.



Solution: The response is the natural response since the source is disconnected at t=0. First the natural response of the circuit will be found and then the coefficients will be found using the initial conditions.



Natural response ($v_s(t)$=0).

KVL:

$$1\frac{di(t)}{dt}+2i(t)+5\int i(t)dt=0$$

Take derivative to get rid of integral form

$$\frac{d^2i(t)}{dt^2}+2\frac{di(t)}{dt}+5i(t)=0$$

$$s_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$s_{1,2}=\frac{-2\pm\sqrt{2^2-4.1.5}}{2.1}=\frac{-2\pm j4}{2}$$

j: imaginary; *j.j*=-1

Solution: $i(t)=Ke^{st}$

$$Ke^{st}(s^2+2s+2)=0 \Rightarrow s^2+2s+5=0 \Rightarrow \begin{matrix} s_1=-1+j2 \\ s_2=-1-j2 \end{matrix}$$

Roots are complex!

$$i(t)=K_1e^{(-1+j2)t}+K_2e^{-(1+j2)t}$$

From the initial conditions (K_1 and K_2):

$$i(t) = K_1 e^{(-1+j2)t} + K_2 e^{-(1+j2)t}$$

factoring out e^{-t} :

$$i(t) = e^{-t}(K_1 e^{+j2t} + K_2 e^{-j2t})$$

This can be rewritten in a simpler format.

From Euler's equation: $e^{\pm i\theta} = \cos\theta \pm j\sin\theta$

$$i(t) = e^{-t}\left[K_1(\cos 2t + j\sin 2t) + K_2(\cos 2t - j\sin 2t)\right]$$

$$i(t) = e^{-t}\left[A\cos 2t + B\sin 2t\right]$$

$$A \equiv K_1 + K_2$$
$$B \equiv j(K_1 - K_2)$$

To find K_1 and K_2 coefficients initial conditions (t=0) should be used.

First initial condition i(t=0)=0

$$i(t=0) = 0 = e^{-0}\left[A\cos 2.(0) + B\sin 2(0)\right] = A \Rightarrow A = 0$$

Second initial conditions: B coeff. can be found: $\frac{di(t)}{dt} = e^{-t}\left[-2A\sin 2t + 2B\cos 2t\right] - e^{-t}\left[A\cos 2t + B\sin 2t\right]$

$$\frac{di(t=0)}{dt} = 0(A=0) + e^{-0}\left[-2A\sin 2(0) + 2B\cos 2(0)\right] = 2B$$

KVL equation at t=0

$$1\frac{di(t=0)}{dt}+2i(t=0)-v_C(t=0)=0 \quad \Rightarrow \quad \frac{di(t=0)}{dt}=10$$

$$i(t=0)=0 \qquad v_C(t=0)=10\ V$$

$$\frac{di(t=0)}{dt}=2B \qquad \frac{di(t=0)}{dt}=2B=10 \Rightarrow B=5$$

Exact solution: $i(t)=5e^{-t}\sin 2t \ \ amper$