Ankara University
Engineering Faculty
Department of Engineering Physics

PEN207

Circuit Design and Analysis

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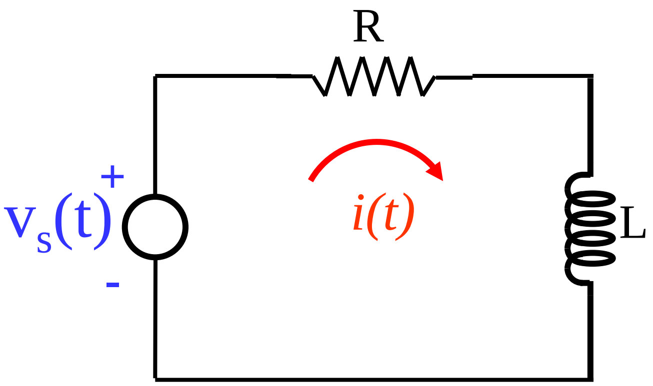
Chapter-3

Circuit Responses (2/2)

Forced Response

Lets look at the circuit below



Kirchhoff's Voltage Law(KVL)

$$L\frac{di(t)}{dt} + Ri(t) = v_s(t)$$

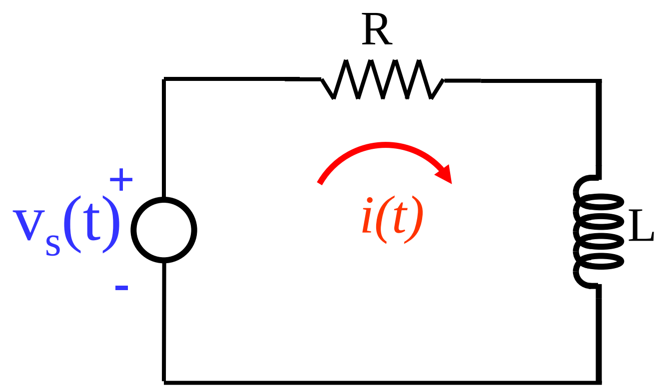
Exact Solution: $i(t) = i_f(t) + i_n(t)$

$i_f(t)$= forced response
$i_n(t)$=natural response

$$\left[L\frac{di_n(t)}{dt} + Ri_n(t)\right] + \left[L\frac{di_f(t)}{dt} + Ri_f(t)\right] = v_s(t)$$

The natural response will become zero $L\frac{di_n(t)}{dt} + Ri_n(t) \to 0$

Forced response will be dominant: $L\frac{di_f(t)}{dt} + Ri_f(t) = v_s(t)$



Kirchhoff's Voltage Law(KVL)

$$L\frac{di_f(t)}{dt}+Ri_f(t)=v_s(t)$$

Source: $v_s(t)$=At

Suppose the solution is in the same form as the source ($v_s(t)$): $i_f(t)=Bt+C$

$$L\frac{d}{dt}(Bt+C)+R(Bt+C)=At$$

Reminder:
Derivative rules

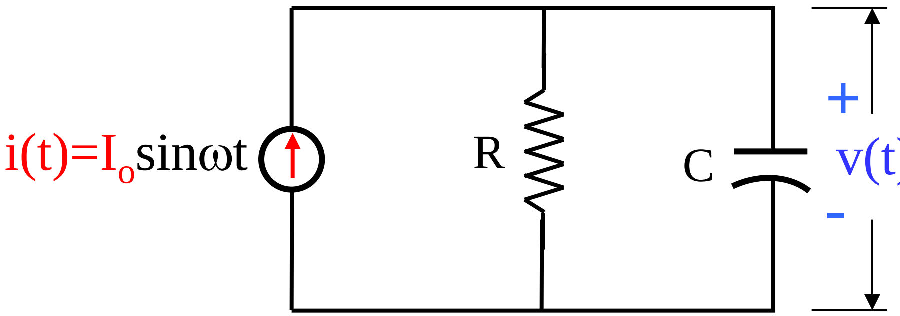
$$y(t)=At^n+B$$
$$\frac{dy(t)}{dt}=nAt^{n-1}$$

$$BL+RBt+RC=At$$

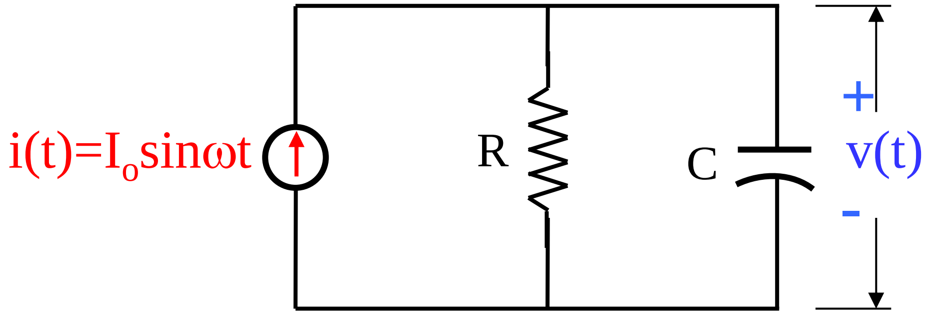
$$(RB)t+(BL+RC)=At \Rightarrow \left.\begin{array}{l} RB=A \\ BL+RC=0 \end{array}\right\} \Rightarrow \begin{array}{l} B=A/R \\ C=-AL/R^2 \end{array}$$

Solution in terms of the constant A: $i_f(t)=\frac{A}{R}t-\frac{AL}{R^2}$

Example-3.5: Find the forced response component $v_f(t)$ of the following circuit's response to sinusoidal current.



Solution:



Kirchhoff's Current Law (KCL):

$$C\frac{dv_f(t)}{dt}+\frac{1}{R}v_f(t)=I_o\sin\omega t$$

To find the forced response we can write $v_f(t)$ as a function of forced excitation and its derivative

$$v_f(t)=A\sin\omega t+B\cos\omega t$$

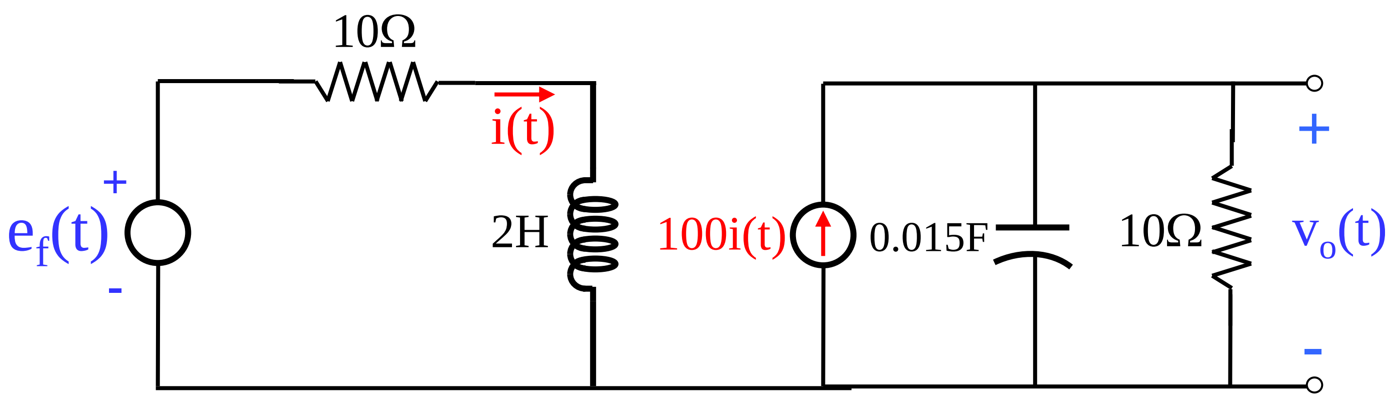
$$C(\omega A\cos\omega t-\omega B\sin\omega t)+\frac{1}{R}(A\sin\omega t+B\cos\omega t)=I\sin\omega t$$

$$\left(C\omega A+\frac{B}{R}\right)\cos\omega t+\left(\frac{A}{R}-C\omega B\right)\sin\omega t=I\sin\omega t \quad\Rightarrow\quad \begin{array}{l} C\omega A+B/R=0 \\ A/R-C\omega B=I \end{array}$$

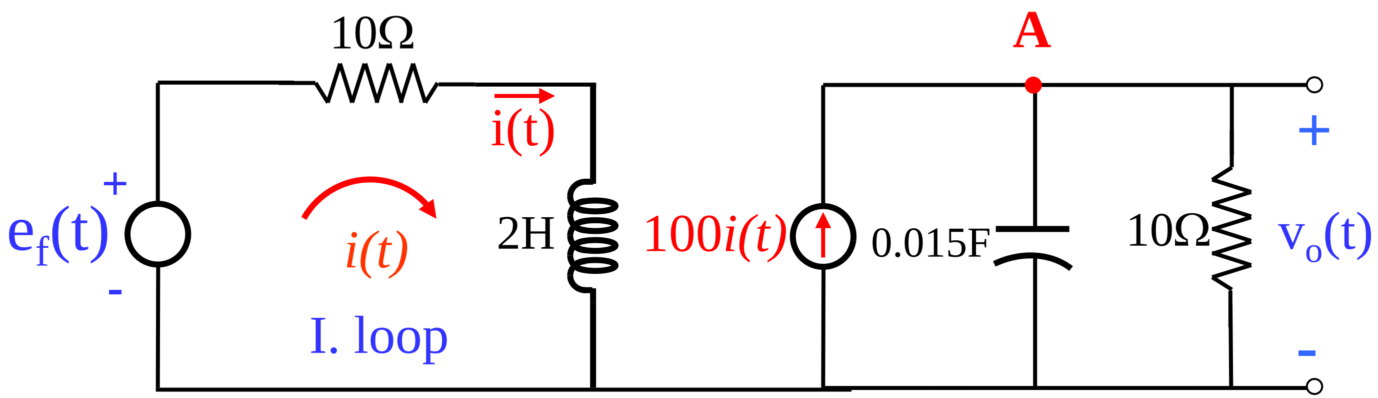
A and B coefficients: $A=I\frac{G}{G^2+(\omega C)^2}$ $\quad B=I\frac{-\omega C}{G^2+(\omega C)^2}$ $\quad G\equiv 1/R$

Solution: $v_f(t)=I\left(\frac{G}{G^2+(\omega C)^2}\sin\omega t-\frac{\omega C}{G^2+(\omega C)^2}\cos\omega t\right)$

Example-3.6: A voltage of $e_f(t)=10e^{-4t}$ mV is applied to the following circuit at time t=0. Find the forced component of the output voltage $v_o(t)$.



Solution: The two equations that define the circuit behavior are the KVL equation applied to the left loop and the KCL equation at junction A



Kirchhoff's Voltage Law (KVL) for the I. loop

$$2\frac{di(t)}{dt}+10i(t)=e_f(t)$$

Kirchhoff's Current Law (KCL) for the junction **A**

$$0.015\frac{dv_o(t)}{dt}+0.1v_o(t)=100i(t)$$

$$2\frac{di(t)}{dt} + 10i(t) = e_f(t)$$

$$0.015\frac{dv_o(t)}{dt} + 0.1v_o(t) = 100i(t)$$

Since the excitation is exponential, we can assume that the response is also in the same form of exponential function:

$$i(t) = Ie^{-4t}$$
$$v_o(t) = E_o e^{-4t}$$

$$-8Ie^{-4t} + 10Ie^{-4t} = E_o e^{-4t}$$

$$-0,06E_o e^{-4t} + 0,1E_o e^{-4t} = 100Ie^{-4t}$$

1 mV=1x10^{-3} V

$$-8Ie^{-4t} + 10Ie^{-4t} = 1\times10^{-2}e^{-4t}$$

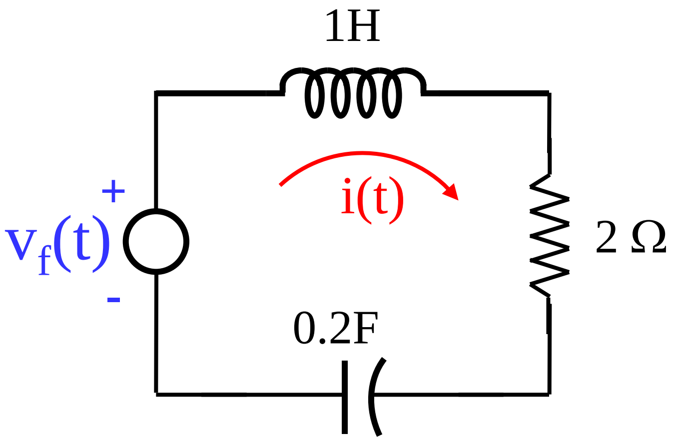
$$-(6\times10^{-4})e^{-4t} + (1\times10^{-3})e^{-4t} = 100Ie^{-4t}$$

I=5x10^{-3} A found.

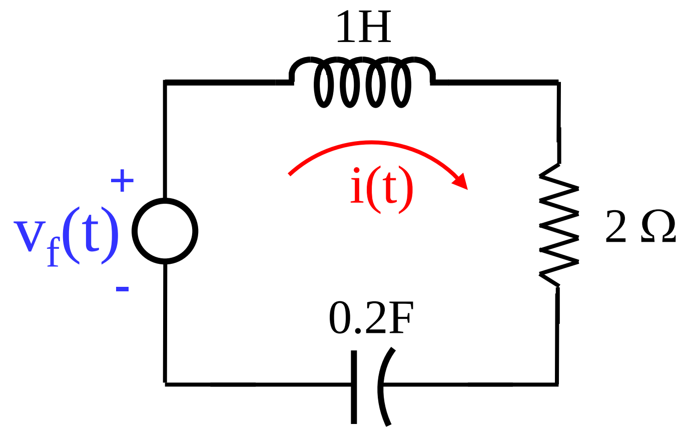
$$E_o = 12.5\ V\ ; \quad v_o(t) = 12.5e^{-4t}\ V$$

The presence of 100i(t) dependent current source in this problem has generated a voltage increase. The voltage at the output is greater than the input voltage. Such circuits are used in electronic amplifiers.

Example-3.7: Find the forced component $i_f(t)$ of current if $v_f(t)=5t^2$ V at the t=0 in the circuit below.



Solution:



$v_f(t)=5t^2$ V

Kirchhoff's Voltage Law (KVL)

$$1\frac{di(t)}{dt}+2i(t)+5\int i(t)dt=v(t) \quad \Rightarrow \quad \frac{d^2i(t)}{dt^2}+2\frac{di(t)}{dt}+5i(t)=\frac{dv(t)}{dt}$$

The forced response must satisfy the above equation. Forced response in this case

$$i_f(t)=At^2+Bt+C$$

(including t^2 and all its derivatives)

$dv_f(t)/dt=10t$

$$2A+2(2At+B)+5(At^2+Bt+C)=10t$$

$$(5A)t^2+(4A+5B)t+(2A+2B+5C)=10t \Rightarrow \begin{cases} 5A=0 \\ (4A+5B)=10 \\ (2A+2B+5C)=0 \end{cases}$$

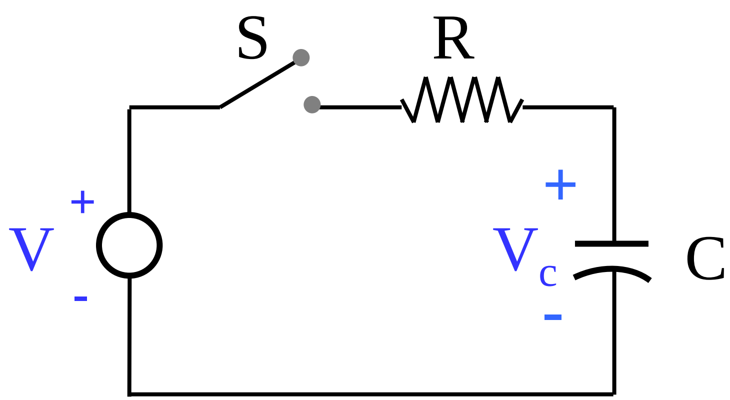
$$A=0; B=2; C=-0.8$$

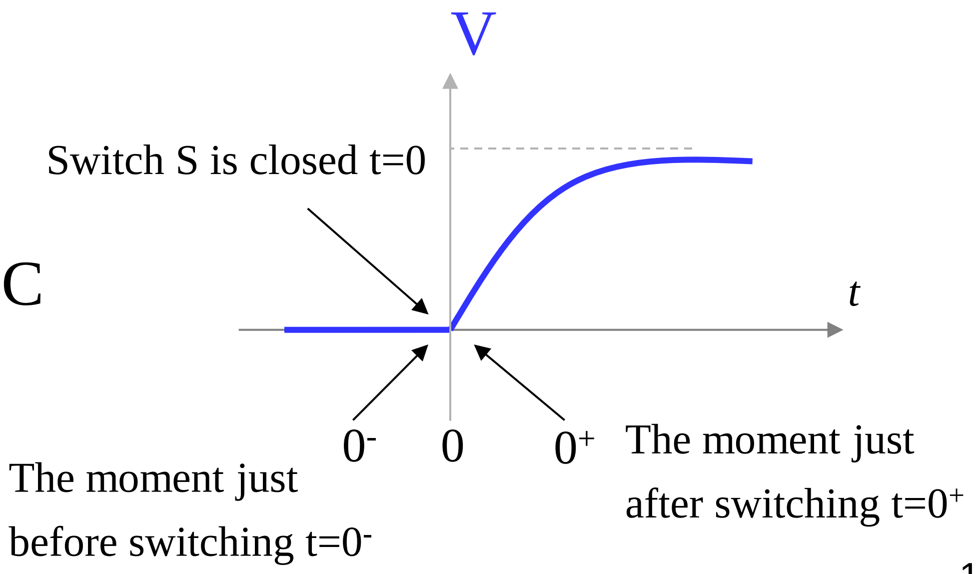
Forced response: $i_f(t)=2t-0.8$ *amper*

Initial Conditions

This section will explain how the initial conditions can be found theoretically. Finding initial conditions based on the Kirchhoff's Laws and the continuity of the capacitance voltage and inductance current. The results are used to calculate the coefficients of the Natural Response terms.

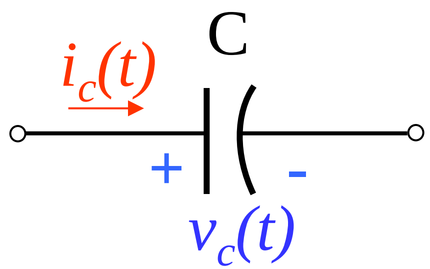
If the moment of switching is assumed t=0, the moment immediately before the switching (0-); the next moment will be displayed as (0+).





Initial Conditions: Capacitor

Conductor:



Current:

$$i_c(t) = C\frac{d\mathrm{v}_c(t)}{dt}$$

Stored energy:

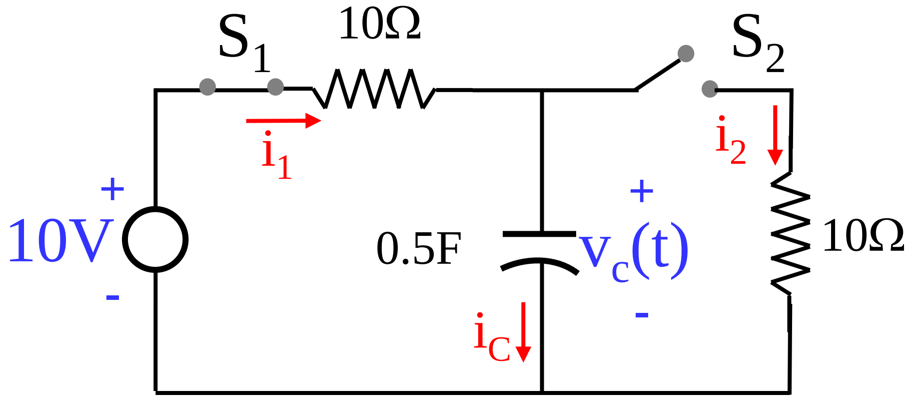
$$W_C = \frac{1}{2}C\mathrm{v}_C^2$$

It indicates that an infinite current must be accompanied by a momentary change in voltage between the capacitance terminals (ie $d\mathrm{v}_C/dt=\infty$)

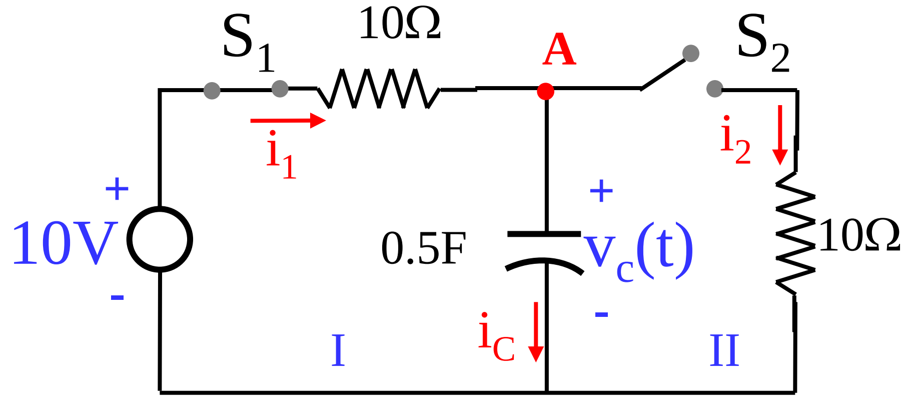
Unless an infinitely large current is supplied, the voltage at the ends of a capacitor cannot be changed suddenly!

$$\mathrm{v}_C(0^-) = \mathrm{v}_C(0^+)$$

Example-3.8: In the circuit below, switch S_1 has been switched off long before t=0. Switch S_2 is also closed at t=0. Immediately after the switch is turned off, find the $v_C(0^+)$ capacitance voltage and the $i_C(0^+)$ capacitance current.



Solution:



When the switches are in the above position (S_1=On; S_2=Off), if the natural response is long enough to charge the capacitor, then the capacitance is loaded up to 10V, and the current through the circuit becomes zero

$$v_C(0^-) = 10\ V \qquad\qquad i_C(0^-) = 0$$

The switching at the time t=0 (switch S_2) connects the second 10Ω resistor to the circuit. From the principle of continuity:

$$v_C(0^+) = v_C(0^-) = 10V$$

In order to find $i_C(0^+)$ KVL and KCL equations at $t=0^+$ are written. These are:

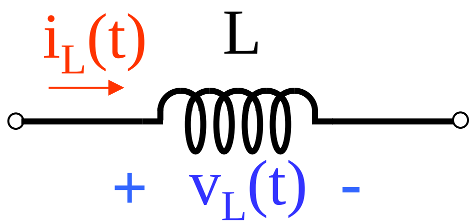
KVL for loop I $\qquad 10 - 10 i_1(0^+) - v_C(0^+) = 0 \quad \Rightarrow \quad i_1(0^+) = 0$

KVL for loop II $\qquad v_C(0^+) - 10 i_2(0^+) = 0 \quad \Rightarrow \quad i_2(0^+) = 1\ A$

KCL for junction **A** $\qquad i_1(0^+) = i_2(0^+) + i_C(0^+)$ then $i_C(0^+) = -1\ A$ found.

Initial Conditions: Inductor

Inductance:



Voltage:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Stored energy:

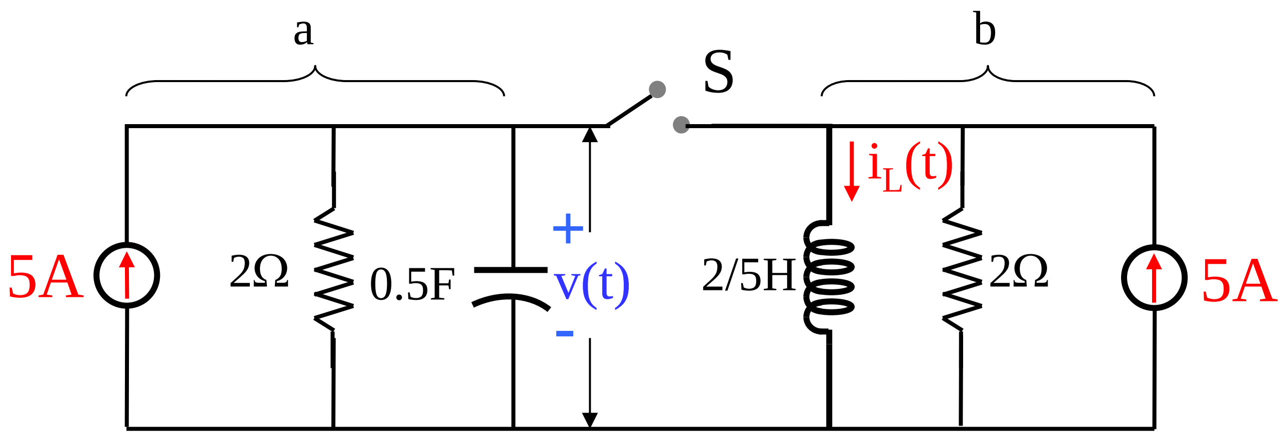
$$W_L = \frac{1}{2}Li^2$$

It indicates that an infinite voltage must be accompanied by a momentary change in current on inductor (i.e. $di_L/dt=\infty$)

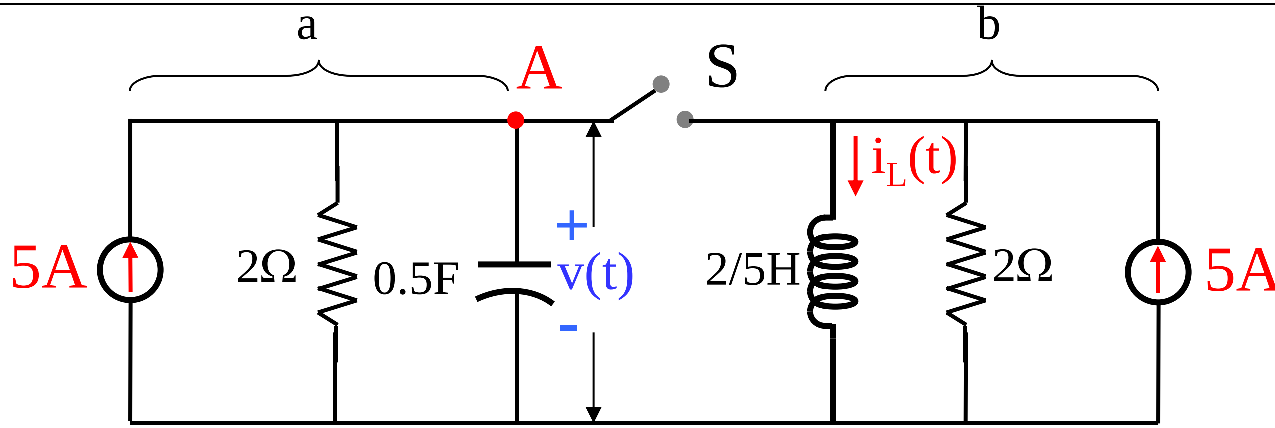
Unless an infinitely large voltage is supplied, the current on a inductance cannot be changed suddenly!

$$i_L(0^-) = i_L(0^+)$$

Example-3.9: In the circuit below, switch S was kept open for a long time before t=0 and was switched on at t=0. Immediately after the switch is closed, find v, dv/dt, i_L and di_L/dt immediately after at t=0^+.



Solution: the capacitance voltage v(t) and the inductance current $i_L(t)$ at t=0 can be found when the switch is closed by applying the continuity principle Kirchhoff's Laws to the circuit.



Before switch is on, parts a and b of the circuit are independent of each other. Since both circuits are supplied with direct current for a long time, their forced response is constant:

In DC case, the coil acts as a short circuit (no current passes through the 2Ω resistor!)

From continuity principle:

$$v(0^-) = 10\ V \qquad i_L(0^-) = 5A$$

$$v(0^+) = 10\ V \qquad i_L(0^+) = 5A$$

If $v(0^+)$=10 V and voltage on inductance is 10V: $\Rightarrow 10V = \frac{2}{5}\frac{di_L(0^+)}{dt}$

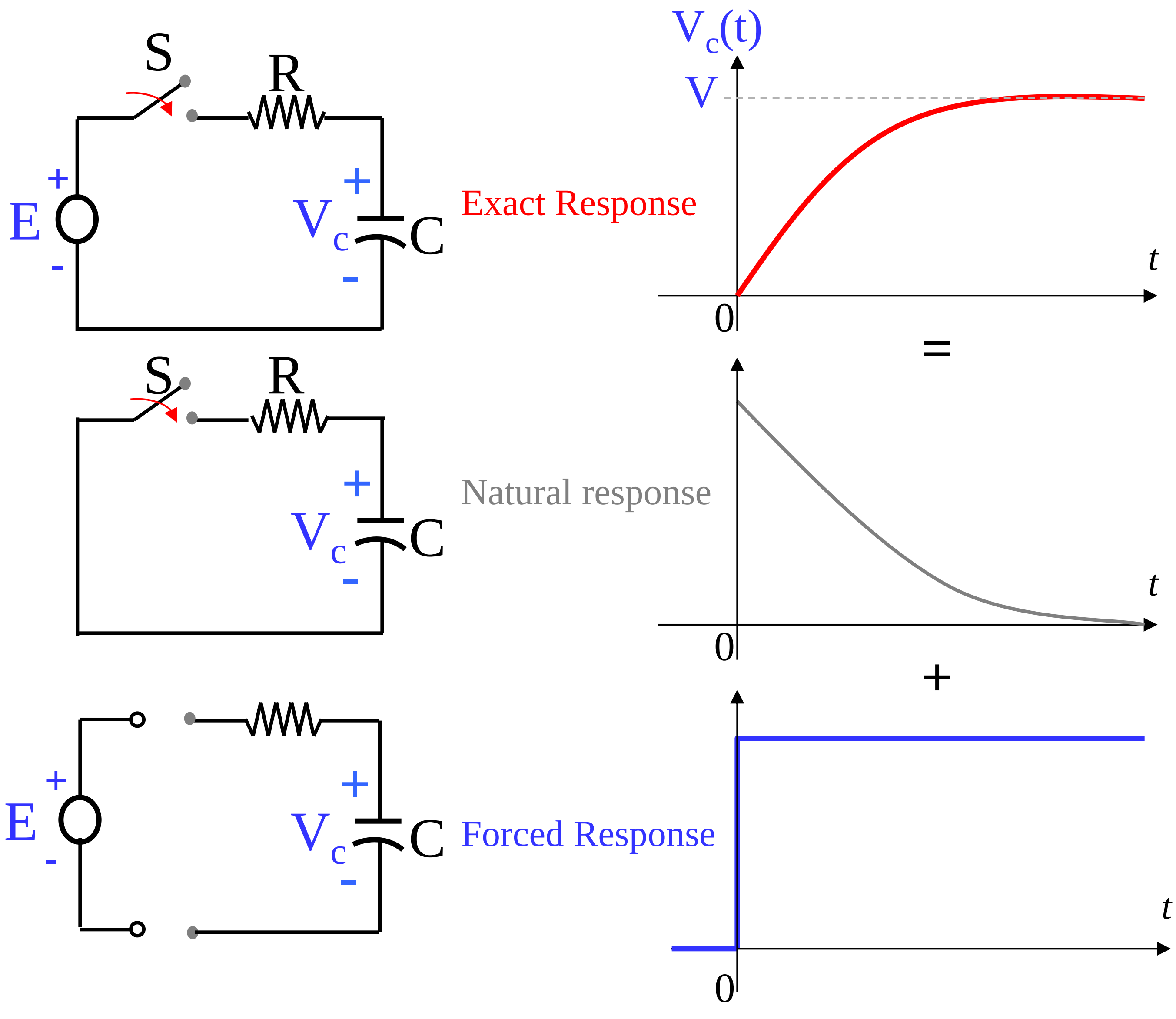
or $$\frac{di_L(0^+)}{dt} = 25A/s$$

Value of dv/dtat t=0 can be found by Kirchhoff's Current Law (KCL) at the juction A

$$+5A - \frac{1}{2}v(0^+) - \frac{1}{2}\frac{dv_C(0^+)}{dt} - i_L(0^+) - \frac{1}{2}v(0^+) + 5A = 0 \Rightarrow \frac{dv(0^+)}{dt} = -10V/s \text{ found.}$$

Full Response

This section deals with the combination of natural and forced responses previously examined separately..



Response of the circuit observed immediately after a sudden change in the circuit (switch-on)

Natural behavior of the circuit with no power supply

Voltage applied by the power supply to the circuit

Full Response

The following steps will be followed in order to systematically examine the full response of a circuit.

1- Write the differential equation for the circuit. If there are integral terms in thedif. equation, it is simplified by taking the derivative of the equation. The equation is arranged on one side of the equation with terms containing independent resources, and terms on the other side of the equation that include circuit parameters and dependent resources.

2- Differantial equatin's roots (s1, s2, s3) are found.

$$K_1 e^{s_1 t} + K_2 e^{s_2 t} + K_3 e^{s_3 t} + \ldots$$

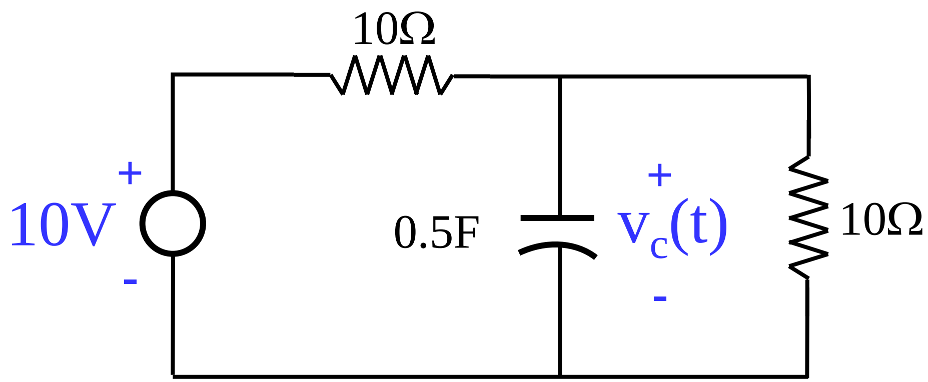
3- **Forced response term is found**.

4- **Forced** and **Natural responses** are added. Sum of these is **Full Response**. K_1, K_2, K_3 etc. coefficients of the natural response are not known for now.

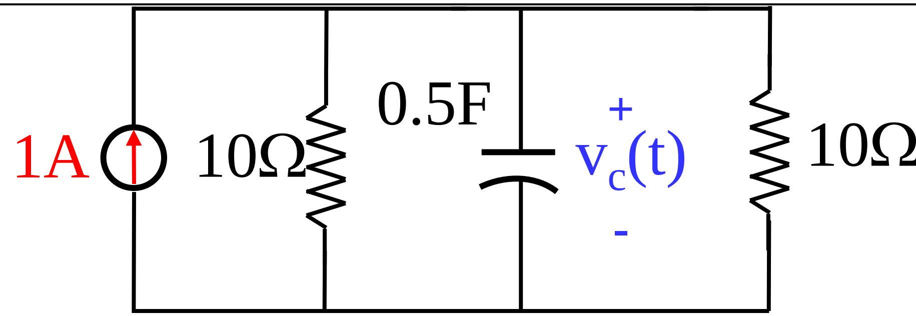
5- Initial conditions are determined. In general, the number of initial conditions is determined by the number of roots of the indicative equation. If the equation has one root, the value of the function at the time t=0, if it has two roots, the function itself and the first derivative value at the time t=0 must be found.

6- Using the initial conditions the value of the K coefficients is found. The full response format should be used in this step.

Example-3.10: Find the value $v_c(t)$ for $t > 0$ in Example-3.8.



Solution: 10Ω and 10V power supply can be converted to an equivalent current source



1- KCL

$$\frac{1}{10}v_C(t)+\frac{1}{2}\frac{dv_C(t)}{dt}+\frac{1}{10}v_C(t)=1A \quad \Rightarrow \quad \frac{dv_C(t)}{dt}+\frac{2}{5}v_C(t)=2$$

2- Natural response (Natural response is found by setting current source off<open circuit>):

$$\frac{dv_C(t)}{dt}+\frac{2}{5}v_C(t)=0$$

Solution:

$$v(t)=Ke^{st} \Rightarrow s+\frac{2}{5}=0 \Rightarrow s=-\frac{2}{5} \Rightarrow v_{cn}(t)=Ke^{-2t/5}$$

3- Since the excitation is a constant (2) forced response component:

$$v_{cf}(t)=A$$

If this value is substituted in the initial differential equation

$$0+\frac{2}{5}A=2 \Rightarrow A=5 \quad \text{found.}$$

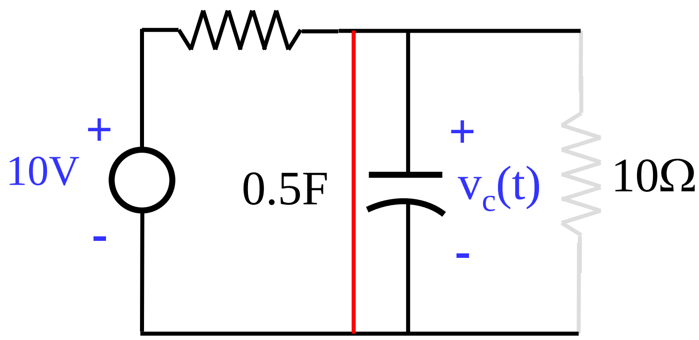
4- Full response: $v_c(t) = v_{cn}(t) + v_{cf}(t) = 5 + Ke^{-2t/5}$

5- Using the result of example-3.8 (from initial conditions) $v_c(0^+) = 10\ V$

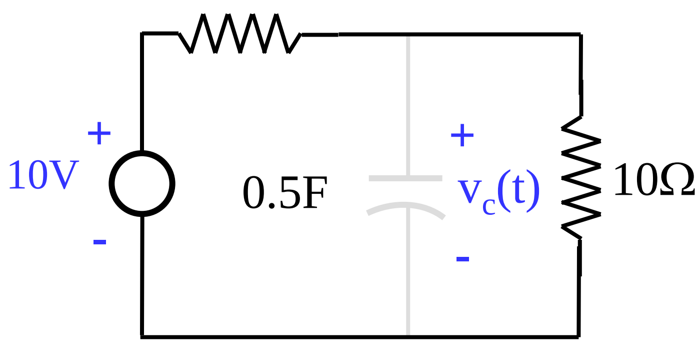
$$10 = 5 + K \quad \Rightarrow \quad K = 5\ V$$

6- By applying initial conditions to the full response solution:

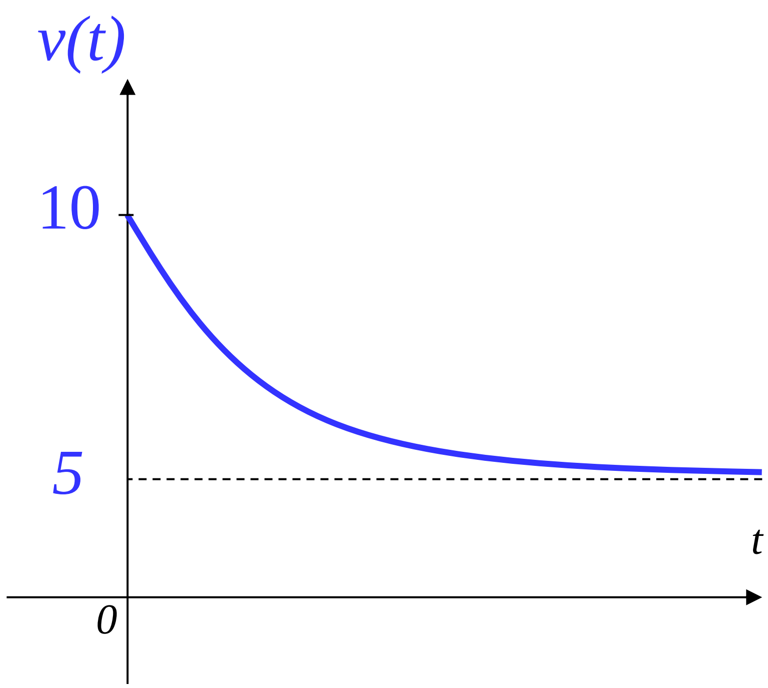
$v_c(t) = 5 + 5e^{-(2/5)t}$ *volt* found.



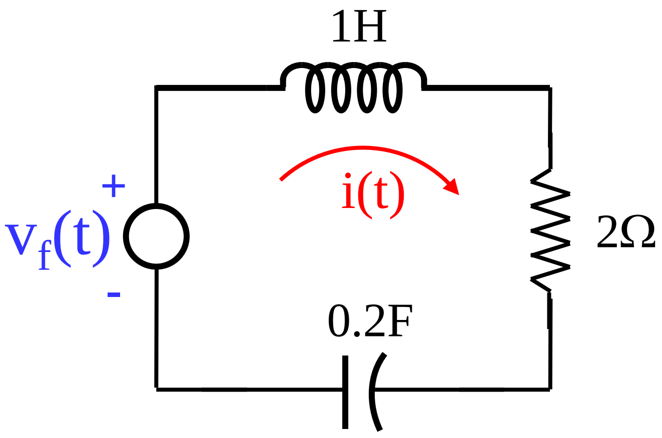
t=0; i=1A; v(t)=10V



t=∞; i=0.5A; v(t)=5V

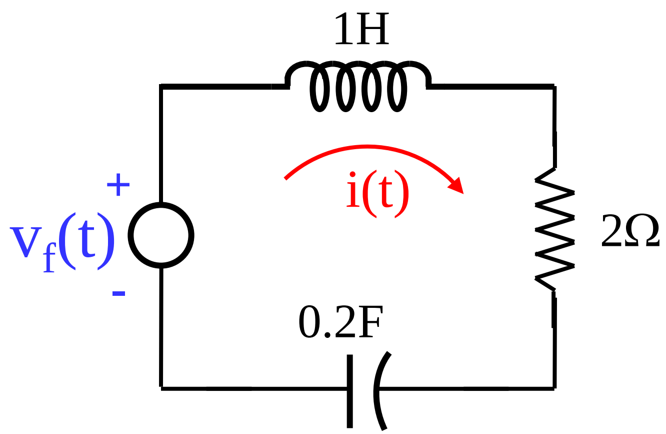


Example-3.11: In Example 3.7, if $v(t)=0$ for $t<0$ and $v(t)=10e^{-4t}$ volts for $t>0$, find the exact response of $i(t)$ for $t>0$.



Solution:

$$v_f(t) = 10e^{-4t}$$



1- KVL equation

$$1\frac{di(t)}{dt} + 2i(t) + 5\int i(t)dt = v_f(t) = 10e^{-4t}$$

Taking the derivative (to get rid of the integral in the equation)

$$\frac{d^2 i(t)}{dt^2} + 2\frac{di(t)}{dt} + 5i(t) = \frac{dv_f(t)}{dt} = -40e^{-4t}$$

2- Natural response equation (v_f=0; Natural response)

$$\frac{d^2 i(t)}{dt^2} + 2\frac{di(t)}{dt} + 5i(t) = 0$$

Proposed solution: $i(t) = Ke^{st}$

Roots:

$$Ke^{st}(s^2 + 2s + 5) = 0 \Rightarrow s^2 + 2s + 5 = 0 \Rightarrow s = -1 \pm j2$$

Natural response component of current: $i_n(t) = e^{-t}(A\cos 2t + B\sin 2t)$

3- The forced component of the response will be in the form of the exponential Ie^{-4t}; when this solution is used in KVL equation

$$(-4)^2 Ie^{-4t} + 2(-4)Ie^{-4t} + 5Ie^{-4t} = -40Ie^{-4t}$$

Then I=-3.08A found. Forced response: $i_f(t) = -3,08e^{-4t}$

4- Full response

$$i(t) = i_n(t) + i_f(t) = -3.08e^{-4t} + e^{-t}(A\cos 2t + B\sin 2t)$$

5- Two initial conditions are needed: $i(0^+)$ ve $di(0^+)/dt$. Before t=0, the power source voltage v(t) has been zero for a long time, so the circuit is in a stand-still condition with all currents and voltages going to zero.

$$i(0^-) = 0; v_c(0^-) = 0 \qquad \text{ve} \qquad i(0^+) = 0; v_c(0^+) = 0$$

In this case, the first initial condition is i(0+)=0. Second one can be found from the KVL equation:

$$\frac{di(0^+)}{dt} + 2i(0^+) + v_V(0^+) = v_V(0^+) = 10 \quad \Rightarrow \quad \frac{di(0^+)}{dt} = 10$$

6- From $i(0^+)=0$ condition

$$i(0^+) = 0 = -3,08 + A \Rightarrow A = 3,08$$

then

$$i(t) = -3,08e^{-4t} + e^{-t}(3,08\cos 2t + B\sin 2t)$$

CoefficientB

$$\frac{di(t)}{dt} = 12,32e^{-4t} + e^{-t}(-6,16\sin 2t + 2B\cos 2t) - e^{-t}(3,08\cos 2t + B\sin 2t)$$

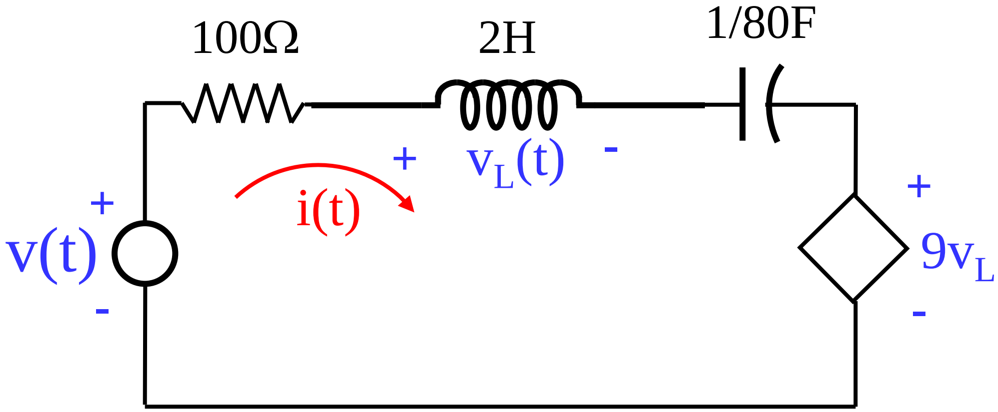
With the use of the initial condition $\frac{di(0^+)}{dt} = 10$

$$\frac{di(0^+)}{dt} = 10 = 12.32 + 2B - 3.08 \Rightarrow B = 0.38$$ is found.

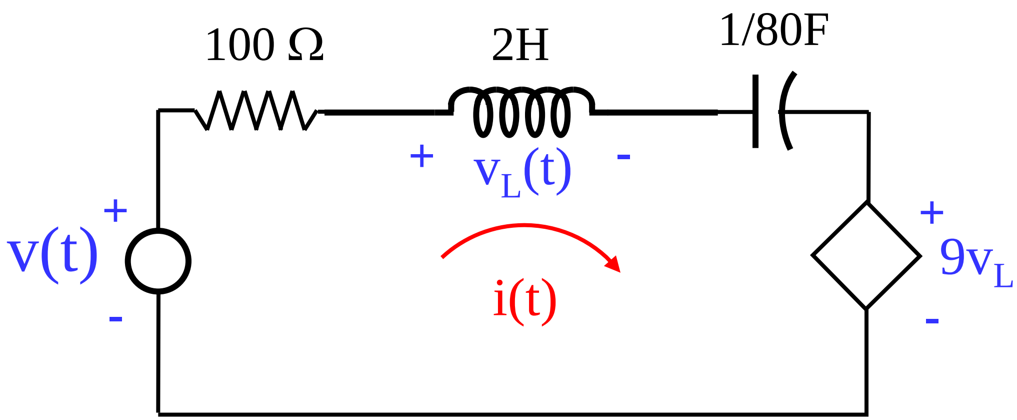
Full response:

$$i(t) = -3.08e^{-4t} + e^{-t}(3.08\cos 2t + 0.38\sin 2t) \;\; amper$$

Example-3.12: The voltage of the power source v(t) in the circuit below is zero for times for t<0 is zero and 100V for t>0. Find the current for t>0.



Solution:



1- KVL

$$100i(t)+2\frac{di(t)}{dt}+80\int i(t)dt=v(t)-9v_L(t)$$

v(t)=100V and inductance's volt-ampere relationship used in the voltage of the dependent voltage source

$$9v_L(t)=9\left(2\frac{di(t)}{dt}\right)$$

$$100i(t)+(2+18)\frac{di(t)}{dt}+80\int i(t)dt=100 \qquad \text{obtained.}$$

if the second derivative is taken $\frac{d^2i(t)}{dt^2}+5\frac{di(t)}{dt}+4i(t)=0$

2- Since the right side of the above equation is zero, this equation is a non-forced equation. Solution of this equation (Natural response):

$$s^2+5s+4=0 \Rightarrow s=-1 \; ve \; s=-4 \qquad \Rightarrow \qquad i_n(t)=K_1e^{-t}+K_2e^{-4t}$$

3- Forced response is zero.

4- Full response:

$$i_n(t)=K_1e^{-t}+K_2e^{-4t}$$

5- Two initial conditions are needed: i(0^+) ve di(0^+)/dt. Since the circuit has been decayed for a long time, there is no stored energy at t=0^-. So,

$$i(0^+)=i(0^-)=0 \qquad \text{and} \qquad v_c(0^+)=v_c(0^-)=0$$

The second initial condition is found from the continuity equation and the value of the KVL at t=0^+

$$100i(0^+)+20\frac{di(0^+)}{dt}+v_C(0^+)=100 \qquad \Rightarrow \qquad \frac{di(0^+)}{dt}=5 \quad \text{found.}$$

6- From boundry conditions

$$i(0) = K_1 + K_2 = 0$$

Then

$$\frac{di(0)}{dt} = -K_1 - 4K_2 = 5 \Rightarrow K_1 = -K_2 = 5/3$$

Full response:

$$i(t) = \frac{5}{3}(e^{-t} - e^{-4t}) \ \ amper$$

found.