CHAPTER 2. FIRST ORDER DIFFERENTIAL EQUATIONS

2.6. First Order Linear Differential Equations

Definition. A differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

is called a first order linear differential equation.

Let us write equation (1) in the differential form

$$(p(x)y - q(x))\,dx + dy = 0 \tag{2}$$

It is clear that equation (2) is not exact, but it can be found integrating factor as

$$\lambda(x) = e^{\int p(x)dx}.$$

Multiplying (1) by $\lambda(x)$, we get

$$\frac{d}{dx}\left[e^{\int p(x)dx}y\right] = e^{\int p(x)dx}q(x)$$

Integrating this equation we get

$$y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + c\right]$$

or

$$y(x) = \frac{1}{\lambda(x)}\left[\int \lambda(x)q(x)dx + c\right]. \tag{3}$$

Example. Solve the following differential equations.
1)

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

Solution. We observe that the equation is linear with $p(x) = \dfrac{2x+1}{x}$ and $q(x) = e^{-2x}$. The integrating factor is obtained as

$$\lambda(x) = e^{\int \frac{2x+1}{x}dx} = e^{2x+\ln x} = xe^{2x}.$$

So, from the formula (3), we have

$$y(x) = \frac{e^{-2x}}{x}\int xdx = \frac{e^{-2x}}{x}\left(\frac{x^2}{2} + c\right) = e^{-2x}\left(\frac{x}{2} + \frac{c}{x}\right).$$

2)

$$(x^2+1)dy = x(1-4y)dx,\ y(2) = 1.$$

3)

$$y^2dx + (3xy-1)dy = 0.$$

2.7. Bernoulli Equation

Definition. An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,\ n \in \mathbb{R} \tag{1}$$

is called Bernoulli differential equation. Note that if $n = 0$, then equation (1) is linear; if $n = 1$, then equation (1) is seperable.

Theorem. Suppose $n \neq 0, 1$. Then the transformation $z = y^{1-n}$ reduces the Bernoulli equation to a linear equation

$$\frac{dy}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Example. Solve the following differential equations.

1)

$$\frac{dy}{dx} + y = xy^3$$

Solution. It is clear that the given equation is Bernoulli with $n = 3$. Applying the transformation $z = y^{-2}$, we get

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dz}{dx}.$$

Substituting $z = y^{-2}$ and $\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dz}{dx}$ into given differential equation we obtain a first order linear differential equation

$$\frac{dz}{dx} - 2z = -2x.$$

Integrating factor for the last equation is obtaines as

$$\lambda(x) = e^{\int -2dx} = e^{-2x}.$$

Multiplying the linear equation by $\lambda(x) = e^{-2x}$, we get

$$\frac{d}{dx}\left[e^{-2x}z\right] = -2xe^{-2x}.$$

Integrating last equation we obtain

$$e^{-2x}z = xe^{-2x} + \frac{1}{2}e^{-2x} + c$$

or

$$z = x + \frac{1}{2} + ce^{2x}.$$

Since $z = y^{-2}$, we obtain the solution of Bernoulli equation as

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$$

2)

$$xy' = 2(y - \sqrt{xy})$$

3)

$$2ydx + x(x^2 \ln y - 1)dy = 0$$