CHAPTER 2. FIRST ORDER DIFFERENTIAL EQUATIONS

2.8. Riccati Equations

Definition. A differential equation of the form

$$\frac{dy}{dx} + p(x)y^2 + q(x)y + r(x) = 0 \tag{1}$$

is called Riccati differential equation.

If $p(x) \equiv 0$, then equation (1) is linear;
If $r(x) \equiv 0$, then equation (1) is Bernoulli;

If p, q and r are constants, then equation (1) is separable

$$\frac{dy}{py^2 + qy + r} = dx.$$

Theorem. If $y_1 = y_1(x)$ is a particular solution of equation (1), then substitution

$$y = y_1(x) + \frac{1}{u(x)}$$

converts the Riccati equation into a first order linear equation in u.

Proof. From the transformation $y = y_1(x) + \frac{1}{u(x)}$, we get

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx}$$

Substituting in equation (1), we have

$$\frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx} + p(x)\left[y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}\right] + q(x)\left[y_1 + \frac{1}{u}\right] + r(x) = 0 \tag{2}$$

Since y_1 is a particular solution of (1) it is satisfied that

$$\frac{dy_1}{dx} + p(x)y_1^2 + q(x)y_1 + r(x) = 0.$$

Writing last equality into (2), we have

$$-\frac{1}{u^2}\frac{du}{dx} + \frac{2y_1p(x)}{u} + \frac{p(x)}{u^2} + \frac{q(x)}{u} = 0.$$

Multiplying by $-u^2$ we obtain linear equation

$$\frac{du}{dx} - \left(2y_1(x)p(x) + q(x)\right)u = p(x).$$

Remark. If two particular solutions y_1, y_2 are known, then the general solution of Riccati equation can be found in terms of an integral:

$$\frac{y-y_1}{y-y_2} = c\exp\left(p(x)\left(y_2(x)-y_1(x)\right)dx\right).$$

Example. Solve the following differential equations.

1)

$$\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x$$

Solution. We observe that the equation is Riccati and a particular solution is $y_1 = 1$. So, from the transformation

$$y = 1 + \frac{1}{u},\ \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

we obtain

$$-\frac{1}{u^2}\frac{du}{dx} = (1-x)\left(1+\frac{2}{u}+\frac{1}{u^2}\right) + (2x-1)\left(1+\frac{1}{u}\right) - x$$

or

$$\frac{du}{dx} + u = x - 1$$

which is a first order linear differential equation. Integrating factor for linear equation is obtained as

$$\lambda(x) = e^x.$$

So, the general solution of linear equation is

$$u(x) = x - 2 + ce^{-x}.$$

Since $y = 1 + \frac{1}{u}$, general solution of given Riccati equation is obtained as

$$y = \frac{x-1+ce^{-x}}{x-2+ce^{-x}}.$$

2)

$$xy' - y^2 + (2x+1)y = x^2 + 2x.$$

3)

$$e^{-x}\frac{dy}{dx} + y^2 - 2ye^x = 1 - e^{2x}.$$

2.9. Substitutions

We note that a differential equation which looks different from any of those that we have studied, may be solved easily by a change of variables. However, we can not give any rule.

Example. Solve the following differential equations.

1)

$$x\frac{dy}{dx} - y = \frac{x^3}{y}e^{y/x}$$

Solution. Let $v = \dfrac{y}{x}$. So, we have

$$\frac{dv}{dx} = \frac{xy' - y}{x^2} = \frac{y'}{x} - \frac{y}{x^2}$$

or

$$\frac{dy}{dx} = x\frac{dv}{dx} + \frac{dy}{dx}.$$

Hence, given equation becomes

$$x^2\frac{dv}{dx} = \frac{x^3}{y}e^{y/x}$$

or

$$\frac{dv}{dx} = \frac{e^v}{v}$$

which is a separable equation. Integration by parts yields

$$-ve^{-v} + e^{-v} = x + c.$$

Since $v = \dfrac{y}{x}$, we obtain the solution of given equation as

$$e^{-y/x}\left(1 + \frac{y}{x}\right) = x + c.$$

2)

$$y' = y - x - 1 + (x - y + 2)^{-1}$$

3)

$$\frac{dy}{dx} + \tan x \tan y \ln(\sin y) = 0.$$