CHAPTER 4. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.1. Basic Theory

Definition 1. A linear ordinary differential equation of order n in the dependent variable y and in the independent variable x is in the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \qquad (1)$$

where a_0 is not identically zero.

If $F(x)$ is identically zero, then equation (1) reduces to

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (2)$$

Equation (2) is called homogeneous equation associated with (1).

Example 1) The equation

$$\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + y = \sin x$$

is a third order variable cofficient nonhomogeneous linear differential equation.

The equation

$$\frac{dy^3}{dx^3} + 3\frac{dy}{dx} - 2y = 0$$

is a third order constant coefficient homogeneous linear differential equation.

Theorem 1. Consider the nth order linear differential equation (1). Let x_0 be any point of the interval $[a, b]$ and $c_1, c_2, ..., c_n$ be n arbitrary real constants. If $a_0(x) \neq 0$ for every $x \in [a, b]$, then there exits a unique solution f such that

$$f(x_0) = c_1, \; f'(x_0) = c_2, ..., f^{(n-1)}(x_0) = c_n$$

and this solution is defined over the interval $[a, b]$.

Example 2. Consider the initial value problem

$$\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + x^2y = e^x; \; y(1) = 1, y'(1) = 2; \; y'''(1) = 1.$$

The coefficients $1, \; 2x, \; x^2$ and the nonhomogeneous term e^x are continuous for all $x \in (-\infty, \infty)$. Moreover the point $x_0 = 1 \in (-\infty, \infty)$. So, by Theorem 1given initial value problem has a unique solution which is defined on $(-\infty, \infty)$.

Corollary 1. Let f be a solution of the nth order homogeneoue linear differential equation (2) such that

$$f(x_0) = f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0, \; x_0 \in [a, b].$$

Then $f(x) \equiv 0$ for all x on $[a, b]$.

Example 3. Let us consider the differential equation

$$\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + x^2y = 0$$

with

$$y(0) = y'(0) = y''(0) = 0$$

By Corollary 1, the unique solution of this initial value problem is $y \equiv 0$.

Definition 2. If $f_1, f_2, ..., f_m$ are given functions and $c_1, c_2, ..., c_m$ are constants, then the expression

$$c_1f_1 + c_2f_2 + ... + c_mf_m$$

is called a linear combination of $f_1, f_2, ..., f_m$.

Theorem 2. Any linear combination of solutions of the homogeneous linear differential equation (2) on $[a, b]$ is also solution on $[a, b]$.

Proof. Let us define

$$f(x) = \sum_{i=1}^{m} c_if_i.$$

Then we have

$$L(D)\sum_{i=1}^{m} c_if_i = \sum_{i=1}^{m} c_iL(D)(f_i) = 0.$$

Example 4. It is easy to see that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation

$$y'' + 4y = 0.$$

By Theorem 2

$$c_1 \cos x + c_2 \sin x$$

is also solution.

Definition 3. If there exist constant $c_1, c_2, ..., c_n$ not all zero such that

$$c_1f_1(x) + c_2f_2(x) + ... + c_nf_n(x) = 0$$

for all $x \in [a, b]$, then the functions $f_1, f_2, ..., f_n$ are called linearly dependent on $[a, b]$.

If the relation

$$c_1f_1(x) + c_2f_2(x) + ... + c_nf_n(x) = 0$$

implies that $c_1 = c_2 = ... = c_n = 0$, then $f_1, f_2, ..., f_n$ are called linearly independent.

Example 5. The functions $\{1, x, x^2\}$ are linearly independent since

$$c_1 + c_2 x + c_3 x^2 = 0$$

implies that $c_1 = c_2 = c_3 = 0$.

The functions $\{e^x, -2e^x\}$ are linearly dependent since the relation

$$c_1 e^x + c_2(-2e^x) = 0$$

is also satisfied when $c_1 \neq 0$ and $c_2 \neq 0$. For example, we can take $c_1 = 2$ and $c_2 = 1$.

Definition 4. Let $f_1, f_2, ..., f_n$ be real, $(n-1)$ times differentiable functions on $[a, b]$. The determinant

$$\begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions $f_1, f_2, ..., f_n$ and it is denoted by $W(f_1, f_2, ..., f_n)(x)$.