4.4. An Operator Method

In this section we consider nonhomogeneous linear differential equations of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+y=f(x), \tag{1}$$

where a_0 is not identically zero and $a_0, a_1, ...a_n$ are real constants.

If $f(x)$ is a UC function, then operator method can be used to find a particular solution of equation (1).

By using the linear differential operator

$$L(D)=a_0D^n+a_1D^{n-1}+...+a_{n-1}D+a_n$$

equation (1) can be written as

$$L(D)y=f(x).$$

So, if y_p is a particular solution of equation (1), then y_p should be satisfy the equation

$$y_p=\frac{1}{L(D)}f(x).$$

Theorem 1. Let a be a real number.
(i) Then

$$\frac{1}{L(D)}\{e^{ax}\}=e^{ax}\frac{1}{L(a)},\ \text{if } L(a)\neq 0;$$

(ii) If $L(a)=0$, then

$$\frac{1}{L(D)}\{e^{ax}\}=e^{ax}\frac{1}{L(D+a)}.$$

Theorem 2. Let $f(x)$ be a polynomial with degree m. Then

$$\frac{1}{L(D)}\{f(x)\}=\left(\sum_{i=0}^{m}c_iD^i\right)(f(x)).$$

Theorem 3. For $m\in\mathbb{R}$

$$\frac{1}{L(D)}\{e^{mx}F(x)\}=e^{mx}\frac{1}{L(D+m)}\{F(x)\}.$$

Theorem 4. Assume that $a \in \mathbb{R}$ and $L(-a^2) \neq 0$. Then

$$\begin{aligned}
\frac{1}{L(D^2)}\{\sin(ax+b)\} &= \frac{1}{L(-a^2)}\{\sin(ax+b)\} \\
\frac{1}{L(D^2)}\{\cos(ax+b)\} &= \frac{1}{L(-a^2)}\{\cos(ax+b)\}
\end{aligned}$$

Remark 1. If $L(-a^2) = 0$, then use the equality

$$e^{iax} = \cos ax + i \sin ax$$

then apply Theorem 1.

Example 1. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = e^x$$

Solution. The characteristic equation of the corresponding differential equation is

$$m^2 - 2m - 3 = 0.$$

So, the roots are $m_1 = 3,\ m_2 = -1$ and the complementary function is

$$y_c = c_1 e^{3x} + c_2 e^{-x}.$$

$$y_p = \frac{1}{D^2 - 2D - 3} e^x.$$

Since $\frac{1}{L(1)} \neq 0$, by Theorem 1, we have

$$y_p = -\frac{1}{4} e^x.$$

So, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{4} e^x.$$

Example 2. Find a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 1 - x$$

Solution.

$$\begin{aligned}
y_p &= \frac{1}{D^2 + 5D - 2}(1-x) \\
&= -\frac{1}{2}\frac{1}{1 - \frac{D^2+5D}{2}}(1-x) \\
&= -\frac{1}{2}\left(1 + \frac{D^2+5D}{2} + \dots\right)(1-x) \\
&= -\frac{1}{2}\left(1 - x - \frac{5}{2}\right) \\
&= \frac{x}{2} + \frac{3}{4}.
\end{aligned}$$

Example. Find the general solutions of following differential equations.

1)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x + 1$$

2)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = xe^x$$

3)

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x + \cos 3x.$$