2. FIRST-ORDER DIFFERENTIAL EQUATIONS

2.1 SEPARABLE EQUATIONS

We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables.

Because the method in this section and many techniques for solving differential equations involve integration, you **are urged to refresh your memory on important formulas** $(\int du/u)$and techniques (such as integration by parts) by consulting a calculus text.

SOLUTION BY INTEGRATION Consider the first-order differential equation $dy/dx = f(x, y)$. When f does not depend on the variable y, that is, $f(x, y) = g(x)$, the differential equation

$$\frac{dy}{dx} = g(x) \tag{1}$$

can be solved by integration. If $g(x)$ is a continuous function, then integrating both sides of (1) gives $y = \int g(x)\,dx = G(x) + c$, where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$. For example, if $dy/dx = 1 + e^{2x}$, then its solution is $y = \int (1 + e^{2x})\,dx$ or $y = x + \frac{1}{2}e^{2x} + c$.

Definition 2.1

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable equations** or to have **separable variables.**

For example, the equations

$$\frac{dy}{dx} = y^2xe^{3x+4y} \qquad \text{and} \qquad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively. In the first equation we can factor $f(x, y) = y^2xe^{3x+4y}$ as

$$f(x, y) = y^2xe^{3x+4y} = \underset{g(x)}{(xe^{3x})}\underset{h(y)}{(y^2e^{4y})},$$

but in the second equation there is no way of expressing $y + \sin x$ as a product of a function of x times a function of y.

Observe that by dividing by the function $h(y)$, we can write a separable equation $\frac{dy}{dx} = g(x)h(y)$ as

$$p(y)\frac{dy}{dx} = g(x)$$

where, for convenience, we have denoted $1/h(y)$ by $p(y)$.

A one-parameter family of solutions, usually given implicitly, is obtained by integrating both sides of

$$p(y)dy = g(x)dx$$

as

$$\int p(y)\,dy = \int g(x)\,dx \qquad \text{or} \qquad H(y) = G(x) + c,$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y) = 1/h(y)$ and $g(x)$, respectively.

Informally speaking, one solves separable equations by performing the separation and then integrating each side.

NOTE There is no need to use two constants in the integration of a separable equation, because if we write $H(y) + c_1 = G(x) + c_2$, then the difference $c_2 - c_1$ can be replaced by a single constant c.

In many instances throughout the chapters that follow, we will relabel constants in a manner convenient to a given equation.

For example, multiples of constants or combinations of constants can sometimes be replaced by a single constant.

Solve Questions

2.2. LINEAR EQUATIONS

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a linear first-order equation is an equation that can be expressed in the form

$$(1) \qquad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \ ,$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

For example, the equation

$$x^2 \sin x - (\cos x)y = (\sin x)\frac{dy}{dx}$$

is linear, because it can be rewritten in the form $(\sin x)\frac{dy}{dx} + (\cos x)y = x^2 \sin x$.
However, the equation

$$y\frac{dy}{dx} + (\sin x)y^3 = e^x + 1$$

İs not linear.

Now let's find how to solve the linear differential equations .

We can summarize the method for solving linear equations as follows.

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\,dx\right] .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{\frac{d}{dx}[\mu(x)y]} = \mu(x)Q(x) ,$$

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

Solve Questions