2.3. Exact Equations

Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = y\,dx + x\,dy.$$

In this section we examine first-order equations in differential form $M(x, y)dx + N(x, y)dy = 0$. By applying a simple test to $M \; and \; N$, we can determine whether $M(x, y)dx + N(x, y)\,dy$ is a differential of a function $f(x, y)$. If the answer is yes, we can construct f by partial integration.

DIFFERENTIAL OF A FUNCTION OF TWO VARIABLES If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \tag{1}$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0. \tag{2}$$

In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then (2) gives the first-order DE

$$(2x - 5y)\, dx + (-5x + 3y^2)\, dy = 0. \tag{3}$$

A DEFINITION Of course, not every first-order DE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ corresponds to a differential of $f(x, y) = c$. So for our purposes it is more important to turn the foregoing example around; namely, if we are given a first-order DE such as (3), is there some way we can recognize that the differential expression $(2x - 5y)\,dx + (-5x + 3y^2)\,dy$ is the differential $d(x^2 - 5xy + y^3)$? If there is, then an implicit solution of (3) is $x^2 - 5xy + y^3 = c$. We answer this question after the next definition.

Definition (Exact Equation)

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example,

$$x^2y^3\,dx + x^3y^2\,dy = 0$$

is an exact equation, because its left-hand side is an exact differential:

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy.$$

Theorem (Criterion for an Exact Differential)

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

METHOD OF SOLUTION Given an equation in the differential form $M(x, y)\,dx + N(x, y)\,dy = 0$, determine whether the equality in (4) holds. If it does, then there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x, y).$$

We can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)\,dx + g(y), \tag{5}$$

where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate (5) with respect to y and assume that $\partial f/\partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y) = N(x, y).$$

This gives

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx. \tag{6}$$

Finally, integrate (6) with respect to y and substitute the result in (5). The implicit solution of the equation is $f(x, y) = c$.

Solve Questions

2.4. Integrating Factors

If we take the standard form for the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x) \ ,$$

and rewrite it in differential form by multiplying through by dx, we obtain

$$\left[P(x)y - Q(x)\right]dx + dy = 0 \ .$$

This form is certainly not exact, but it becomes exact upon multiplication by the integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

We have

$$\left[\mu(x)P(x)y - \mu(x)Q(x)\right]dx + \mu(x)dy = 0$$

as the form, and the compatibility condition is precisely the identity $\mu(x)P(x) = \mu'(x)$

This leads us to generalize the notion of an integrating factor.

Definition (Integrating Factor)

Integrating Factor

Definition 3. If the equation

(1) $M(x, y)dx + N(x, y)dy = 0$

is not exact, but the equation

(2) $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$,

which results from multiplying equation (1) by the function $\mu(x, y)$, *is* exact, then $\mu(x, y)$ is called an **integrating factor**[†] of the equation (1).

Example 1 Show that $\mu(x, y) = xy^2$ is an integrating factor for

(3) $(2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0$.

Use this integrating factor to solve the equation.

How do we find an integrating factor?

Summary

Special Integrating Factors

Theorem 3. If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

(8) $$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right]$$

is an integrating factor for equation (1).

If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

(9) $$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right]$$

is an integrating factor for equation (1).

Method for Finding Special Integrating Factors

If $M\,dx + N\,dy = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} . \tag{10}$$

If (10) is a function of just x, then an integrating factor is given by formula (8). If not, consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} . \tag{11}$$

If (11) is a function of just y, then an integrating factor is given by formula (9).

Solve Questions