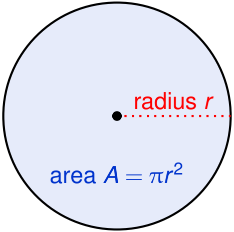
Functions

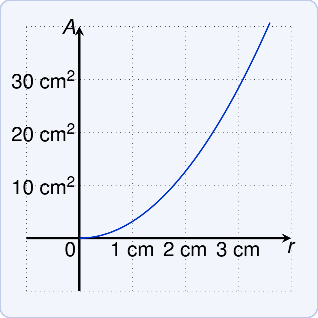
Example

The area A of a circle depends on its radius r. The rule is

$$A = \pi r^2$$

We say that A is a **function** of r.

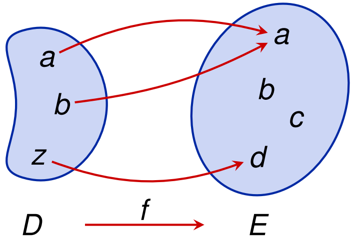




Functions

A **function** f from D to E is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Visualizing functions as **arrow diagrams**:



This example

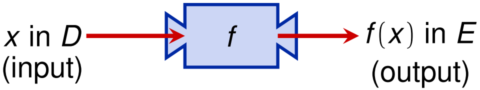
- domain $D = \{ a, b, z \}$
- $E = \{ a, b, c, d \}$
- $f(a) = a$
- $f(b) = a$
- $f(z) = d$
- range $= \{ a, d \}$

Terminology:

- $f(x)$ is the value of f at x
- **domain** of f is the set D
- **range** of f is the set of all possible values $f(x)$ for x in D

Functions as Machines

A function as a **machine**:



- **domain** = set of all possible inputs
- **range** = set of all possible outputs

Example

Square $f(x) = x^2$:

- domain = $\mathbb{R}$
- range = $\{x \mid x \geq 0\}$ = $[0, \infty)$

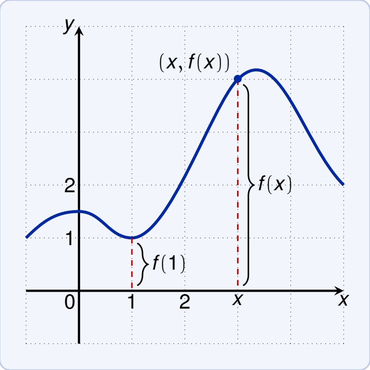
Square root $f(x) = \sqrt{x}$ (over real numbers):

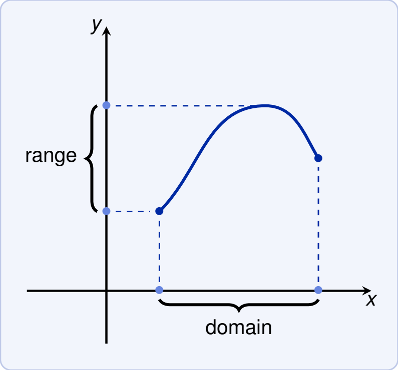
- domain = $\{x \mid x \geq 0\}$ = $[0, \infty)$
- range = $\{x \mid x \geq 0\}$ = $[0, \infty)$

Visualizing Functions as Graphs

The **graph** of a function f is the set of pairs $\{\ (x, f(x)) \mid x \in D\ \}$

- set of all points (x, y) in the coordinate plane
  such that $y = f(x)$ and x is in the domain





Functions: Examples

What is $f(3)$?

- $f(3) = 4$

What is the domain and range of this function?

- domain $= \{x \mid 1 \leq x \leq 4\} = [1, 4]$
- range $= \{y \mid 1 \leq x \leq 5\} = [1, 5]$

Functions: Examples

What is the domain and range of $f(x) = \sqrt{x+2}$?

- domain $= \{x \mid x \geq -2\} = [-2, \infty)$
- range $= \{y \mid y \geq 0\} = [0, \infty)$

What is the domain of $g(x) = \frac{1}{x^2 - x}$?

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

Thus $g(x)$ is **not** defined if $x = 0$ or $x = 1$. The domain is

$$\{x \mid x \neq 0,\ x \neq 1\}$$

which can also be written as

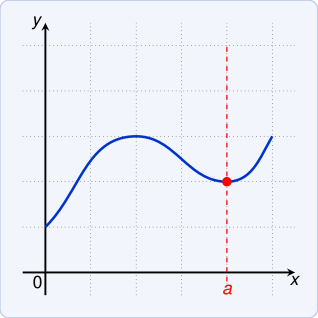
$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

Vertical Line Test

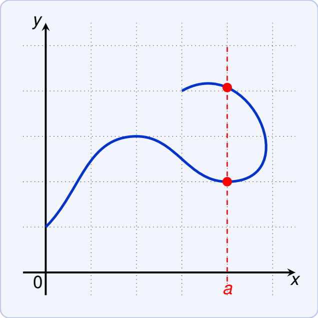
When does a curve represent a function?

Vertical Line Test

A curve in the *xy*-plane represents a function of *x* if and only if no vertical line intersects the curve more than once.



corresponds to a function of *x*



does not correspond to a function of *x*

Representations of Functions

Functions can be represented in four ways:

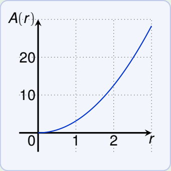
- verbally (a description in words)

Example: $A(r)$ is the area of a circle with radius r.

- numerically (a table of values)

r	1	2	3
$A(r)$	3.14159	12.56637	28.27433

- visually (a graph)



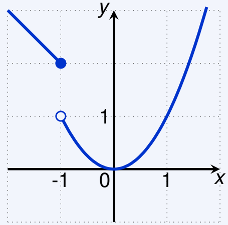
- algebraically (an explicit formula)

$$A(r) = \pi r^2$$

Piecewise Defined Functions

A **piecewise defined** function is defined by different formulas in parts of its domain.

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

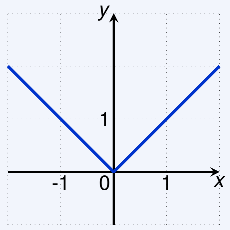


- ● point belongs to the graph
- ○ point is not in the graph

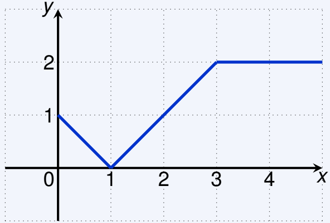
Piecewise Defined Functions: Example

The **absolute value function** $f(x) = |x|$ is piecewise defined:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Piecewise Defined Functions: Example



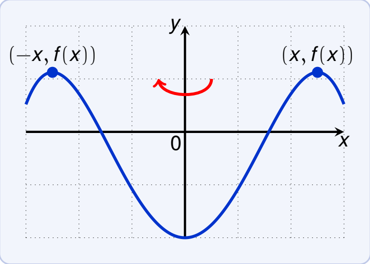
Find a formula for the function f with the graph above.

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x \leq 1 \\ x - 1 & \text{if } 1 < x \leq 3 \\ 2 & \text{if } x > 3 \end{cases}$$

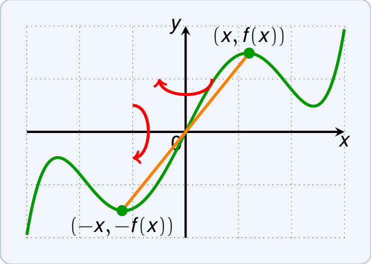
Symmetry

A function f is called

- **even** if $f(-x) = f(x)$ for every x in its domain, and
- **odd** if $f(-x) = -f(x)$ for every x in its domain.



an even function



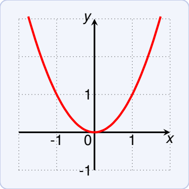
an odd function

- even functions are mirrored around the y-axis
- odd functions are mirrored around the y-axis and x-axis (or mirrored through the point $(0, 0)$)

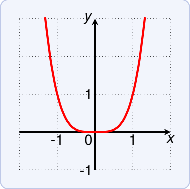
Symmetry

How to remember what is even and odd?

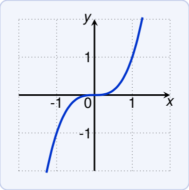
x^2



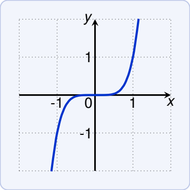
x^4



x^3



x^5



Thick of **power functions** x^n with n a natural number:

- x^n is even if n is even
- x^n is odd if n is odd

Symmetry

Which of the following functions is even?

1. $f(x) = x^5 + x$
2. $g(x) = 1 - x^4$
3. $h(x) = 2x - x^2$

We have:

1. $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

   Thus f is odd.
2. $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

   Thus g is even.
3. $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

   Thus h is neither even nor odd.

Note that:

- The sum of even functions is even (e.g. $1 + x^4$).
- The sum of odd functions is odd (e.g. $x^5 + x$).

Increasing and Decreasing Functions

A function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ and } x_1, x_2 \in I$$

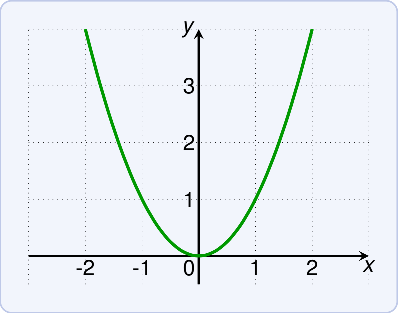
The function is **decreasing** on an interval I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ and } x_1, x_2 \in I$$

This function is:

- increasing on $[0, 3]$
- decreasing on $[3, 4]$
- increasing on $[4, 6]$

Increasing and Decreasing Functions



The function $f(x) = x^2$ is:

- increasing on $[0, \infty)$
- decreasing on $(-\infty, 0]$