Precise Definition of Limits

Recall the definition of limits:

Suppose $f(x)$ is defined close to a (but not necessarily a itself). We write

$$\lim_{x \to a} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches a, is L"

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

The intuitive definition of limits is for some purposes too vague:

- What means 'make $f(x)$ arbitrarily close to L' ?
- What means 'taking x sufficiently close to a' ?

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

Intuitively, when x is close to 3 but $x \neq 3$ then $f(x)$ is close to 5.

How close to 3 does x need to be for $f(x)$ to differ from 5 less than 0.1?

- the distance of x to 3 is $|x - 3|$
- the distance of $f(x)$ to 5 is $|f(x) - 5|$

To answer the question we need to find $\delta > 0$ such that

$$|f(x) - 5| < 0.1 \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

For $x \neq 3$ we have

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 0.1$$

Thus $|f(x) - 5| < 0.1$ whenever $0 < |x - 3| < 0.05$; i.e. $\delta = 0.05$.

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

We have derived

$$|f(x) - 5| < 0.1 \quad \text{whenever} \quad 0 < |x - 3| < 0.05$$

In words this means:

If x is within a distance of 0.05 from 3 (and $x \neq 3$)
then $f(x)$ is within a distance of 0.1 from 5.

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

Similarly, we find

$$|f(x) - 5| < 0.1 \quad \text{whenever} \quad 0 < |x - 3| < 0.05$$
$$|f(x) - 5| < 0.01 \quad \text{whenever} \quad 0 < |x - 3| < 0.005$$
$$|f(x) - 5| < 0.001 \quad \text{whenever} \quad 0 < |x - 3| < 0.0005$$

The distances 0.1, 0.01, ... are called **error tolerance.**

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

Similarly, we find

$$\begin{aligned} |f(x) - 5| < 0.1 &\quad \text{whenever} \quad 0 < |x - 3| < \delta(0.1) \\ |f(x) - 5| < 0.01 &\quad \text{whenever} \quad 0 < |x - 3| < 0.005 \\ |f(x) - 5| < 0.001 &\quad \text{whenever} \quad 0 < |x - 3| < 0.0005 \end{aligned}$$

The distances 0.1, 0.01, $\ldots$ are called **error tolerance.**

We have: $\delta(0.1) = 0.05$

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

Similarly, we find

$$\begin{aligned} |f(x) - 5| < 0.1 \quad &\text{whenever} \quad 0 < |x - 3| < \delta(0.1) \\ |f(x) - 5| < 0.01 \quad &\text{whenever} \quad 0 < |x - 3| < \delta(0.01) \\ |f(x) - 5| < 0.001 \quad &\text{whenever} \quad 0 < |x - 3| < 0.0005 \end{aligned}$$

The distances 0.1, 0.01, $\ldots$ are called **error tolerance.**

We have: $\delta(0.1) = 0.05$, $\delta(0.01) = 0.005$

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x-1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

Similarly, we find

$$|f(x)-5| < 0.1 \quad \text{whenever} \quad 0 < |x-3| < \delta(0.1)$$

$$|f(x)-5| < 0.01 \quad \text{whenever} \quad 0 < |x-3| < \delta(0.01)$$

$$|f(x)-5| < 0.001 \quad \text{whenever} \quad 0 < |x-3| < \delta(0.001)$$

The distances 0.1, 0.01, ... are called **error tolerance.**

We have: $\delta(0.1) = 0.05$, $\delta(0.01) = 0.005$, $\delta(0.001) = 0.0005$

Thus $\delta(\epsilon)$ **is a function of the error tolerance** ϵ!

We need to define $\delta(\epsilon)$ for arbitrary error tolerance $\epsilon > 0$:

$$|f(x)-5| < \epsilon \quad \text{whenever} \quad 0 < |x-3| < \delta(\epsilon)$$

We want $|f(x)-5| = 2|x-3| < \epsilon$. We define $\delta(\epsilon) = \epsilon/2$.

Precise Definition of Limits: Example

$$f(x) = \begin{cases} 2x - 1 & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

We define $\delta(\epsilon) = \epsilon/2$. Then the following holds

$$\text{if} \quad 0 < |x - 3| < \delta(\epsilon) \quad \text{then} \quad |f(x) - 5| < \epsilon$$

In words this means:

> If x is within a distance of $\epsilon/2$ from 3 (and $x \neq 3$)
> then $f(x)$ is within a distance of ϵ from 5.

We can make ϵ arbitrarily small (but greater 0),
and thereby make $f(x)$ arbitrarily close 5.

This motivates the precise definition of limits...

Precise Definition of Limits

Let f be a function that is defined on some open interval that contains a, except possibly on a itself.

$$\lim_{x \to a} f(x) = L$$

if there exists a function $\delta : (0, \infty) \to (0, \infty)$ s.t. for every $\epsilon > 0$:

$$\text{if} \quad 0 < |a - x| < \delta(\epsilon) \quad \text{then} \quad |f(x) - L| < \epsilon$$

In words: No matter what $\epsilon > 0$ we choose,

if the distance of x to a is smaller than $\delta(\epsilon)$ (and $x \neq a$)

then the distance of $f(x)$ to L is smaller than ϵ.

We can make f arbitrarily close to L by taking ϵ arbitrarily small.

Then x is sufficiently close to a if the distance is $< \delta(\epsilon)$.

Precise Definition of Limits

Let f be a function that is defined on some open interval that contains a, except possibly on a itself.

$$\lim_{x \to a} f(x) = L$$

if there exists a function $\delta : (0, \infty) \to (0, \infty)$ s.t. for every $\epsilon > 0$:

$$\text{if} \quad 0 < |a - x| < \delta(\epsilon) \quad \text{then} \quad |f(x) - L| < \epsilon$$

The definition is **equivalent to the one in the book**:

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Precise Definition of Limits

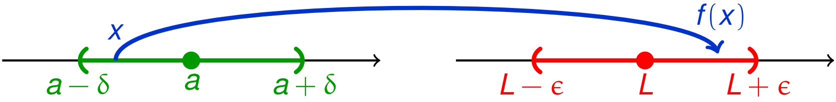
$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Geometric interpretation:

For any small interval $(L - \epsilon, L + \epsilon)$ around L,
we can find an interval $(a - \delta, a + \delta)$ around a
such that f maps all points in $(a - \delta, a + \delta)$ into $(L - \epsilon, L + \epsilon)$.



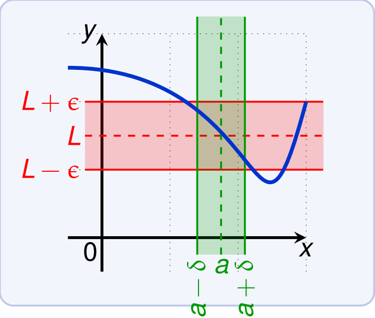
Precise Definition of Limits

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Alternative geometric interpretation:



For every interval I_L around L,

find interval I_a around a

such that

if we restrict the domain of f to I_a, then the curve lies in I_L.

Precise Definition of Limits - Example

Proof that

$$\lim_{x \to 3} (4x - 5) = 7$$

Let $\epsilon > 0$ be arbitrary (the error tolerance).

We need to find δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \epsilon$$

We have

$$\begin{aligned} |(4x - 5) - 7| < \epsilon \quad &\iff \quad |4x - 12| < \epsilon \\ &\iff \quad -\epsilon < 4x - 12 < \epsilon \\ &\iff \quad -\frac{\epsilon}{4} < x - 3 < \frac{\epsilon}{4} \\ &\iff \quad |x - 3| < \frac{\epsilon}{4} \end{aligned}$$

Thus $\delta = \frac{\epsilon}{4}$. If $0 < |x - 3| < \frac{\epsilon}{4}$ then $|(4x - 5) - 7| < \epsilon$.

Precise Definition of Limits - Example

If the next exam will be insanely hard,
then many students will fail.

The words if and then are hugely important!

In exams many students write:

$$0 < |x - 3| < \frac{\epsilon}{4}$$

$$|(4x - 5) - 7| < \epsilon$$

which is wrong.

Correct is:

If $0 < |x - 3| < \frac{\epsilon}{4}$

then $|(4x - 5) - 7| < \epsilon$

Precise Definition of Limits - Example

Find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 1| < \delta \quad \text{then} \quad |(x^2 - 5x + 6) - 2| < 0.2$$

Note that δ is a bound on the distance of x from 1.
Lets say $x = 1 + \delta$. Then

$$\begin{aligned}(x^2 - 5x + 6) - 2 &= (1 + \delta)^2 - 5(1 + \delta) + 4 \\ &= (1 + 2\delta + \delta^2) - (5 + 5\delta) + 4 \\ &= \delta^2 - 3\delta\end{aligned}$$

Thus

$$|(x^2 - 5x + 6) - 2| < 0.2 \quad \iff \quad |\delta^2 - 3\delta| < 0.2$$

Assume that $|\delta| < 1$ (we can make it as small as we want), then:

$$|\delta^2 - 3\delta| \;\leq\; |\delta^2| + |3\delta| \;\leq\; |\delta| + |3\delta| \;\leq\; 4|\delta|$$

Thus: if $4|\delta| < 0.2$ then $|(x^2 - 5x + 6) - 2| < 0.2$.

Hence $\delta = 0.04$ is a possible choice.

Precise Definition of Limits: Example

Let $\lim_{x \to a} f(x) = L_f$ and $\lim_{x \to a} g(x) = L_g$. Prove the sum law:

$$\lim_{x \to a} [f(x) + g(x)] = L_f + L_g$$

Let $\epsilon > 0$ be arbitrary, we need to find δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |(f(x) + g(x)) - (L_f + L_g)| < \epsilon$$

Note that $(f(x) + g(x)) - (L_f + L_g) \;=\; (f(x) - L_f) + (g(x) - L_g)$.

We know that there exists δ_f such that:

$$\text{if} \quad 0 < |x - a| < \delta_f \quad \text{then} \quad |f(x) - L_f| < \epsilon/2$$

and there exists δ_g such that:

$$\text{if} \quad 0 < |x - a| < \delta_g \quad \text{then} \quad |g(x) - L_g| < \epsilon/2$$

We take $\delta = \min(\delta_f, \delta_g)$. If $0 < |x - a| < \delta$ then

$$|f(x) - L_f| < \epsilon/2 \qquad \text{and} \qquad |g(x) - L_g| < \epsilon/2$$

and hence $|(f(x) - L_f) + (g(x) - L_g)| < \epsilon$.

Precise Definition of One-Sided Limits

Left-limit

$$\lim_{x \to a^-} f(x) = L$$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \epsilon$$

Right-limit

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Precise Definition of One-Sided Limits - Example

Right-limit

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Proof that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Let $\epsilon > 0$. We look for $\delta > 0$ such that

$$\text{if} \quad 0 < x < 0 + \delta \quad \text{then} \quad |\sqrt{x} - 0| < \epsilon$$

We have (since $0 < x$)

$$|\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x} < \epsilon \quad \Longrightarrow \quad x < \epsilon^2$$

Thus $\delta = \epsilon^2$. If $0 < x < 0 + \epsilon^2$ then $|\sqrt{x} - 0| < \epsilon$.

Precise Definition of Infinite Limits

Infinite Limit

$$\lim_{x \to a} f(x) = \infty$$

if for every positive number M there is $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad f(x) > M$$

Negative Infinite Limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every negative number M there is $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad f(x) < M$$

Precise Definition of Infinite Limits - Example

Infinite Limit

$$\lim_{x \to a} f(x) = \infty$$

if for every positive number M there is $\delta > 0$ such that

$$\text{if} \quad 0 < |a - x| < \delta \quad \text{then} \quad f(x) > M$$

Proof that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Let M be a positive number. We look for δ such that

$$\text{if} \quad 0 < |0 - x| < \delta \quad \text{then} \quad \frac{1}{x^2} > M$$

We have:

$$\frac{1}{x^2} > M \iff 1 > M \cdot x^2 \iff \frac{1}{M} > x^2 \iff \sqrt{\frac{1}{M}} > |x|$$

Thus $\delta = \sqrt{1/M}$. If $0 < |0 - x| < \sqrt{1/M}$ then $\frac{1}{x^2} > M$.