Continuity

A function f is **continuous** at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

The definition implicitly requires that:

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists

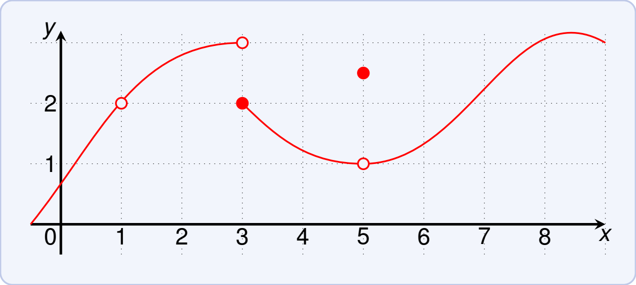
Intuitive meaning of continuous:

- gradual process without interruption or abrupt change
- small changes in x produce only small change in $f(x)$
- graph of the function can be drawn without lifting the pen

A function f is **discontinuous** at a number a if

- f is defined near a (except perhaps a), and
- f is not continuous at a

Continuity: Examples



Where is this graph continuous/discontinuous?

- discontinuous at $x = 1$ since $f(1)$ is not defined
- discontinuous at $x = 3$ since $\lim_{x \to 3} f(x)$ does not exist
- discontinuous at $x = 5$ since $\lim_{x \to 5} f(x) \neq f(5)$

Everywhere else it is continuous.

Continuity: Examples

Where is $\frac{x^2-x-2}{x-2}$ (dis)continuous?

- discontinuous at $x = 2$ since $f(2)$ is undefined
- continuous everywhere else by direct substitution property

Where is

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

(dis)continuous?

- discontinuous at $x = 0$ since $\lim_{x \to 0} f(x)$ does not exist
- continuous everywhere else by direct substitution property

Continuity: Examples

A function f is **continuous form the right** at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

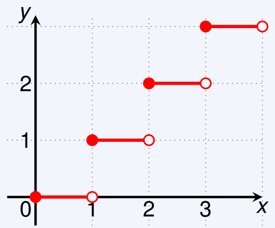
A function f is **continuous form the left** at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Where is $\lfloor x \rfloor$ (dis)continuous?

$$\lfloor x \rfloor = \text{' the largest integer} \leq x \text{ '}$$

- discontinuous at all integers
- left-discontinuous at all integers
  $\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \neq n = f(n)$
- but right-continuous everywhere
  $\lim_{x \to n^+} \lfloor x \rfloor = n = f(n)$



Continuity on Intervals

A function f is **continuous** on an interval if it is continuous on every number in the interval.

If the interval is left- and/or right-closed, then

- At the left-end we are only interested in right-continuity.
- At the right-end we are only interested in left-continuity.

(the values outside of the interval do not matter)

Show that $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

For $-1 < a < 1$ we have by the limit laws:

$$\lim_{x \to a} f(x) = 1 - \sqrt{\lim_{x \to a}(1 - x^2)} = 1 - \sqrt{1 - a^2} = f(a)$$

Similar calculations show

- $\lim_{x \to -1^+} f(x) = 1 = f(-1)$
- $\lim_{x \to 1^-} f(x) = 1 = f(1)$

Therefore f is continuous on $[-1, 1]$.

Continuity: Composition of Functions

If f and g are continuous at a and c is a constant, then the following functions are continuous at a:

1. $f + g$
2. $f - g$
3. $c \cdot f$
4. $f \cdot g$
5. $\frac{f}{g}$ if $g(a) \neq 0$

All of these can be proven from the limit laws!

For example, (1) can be proven as follows:

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$= f(a) + g(a) = (f + g)(a)$$

Thus $f + g$ is continuous at a.

Continuity

These functions are continuous at each point of their domain:

polynomials rationals root functions

(inverse) trigonometric exponential logarithmic

Inverse functions of continuous functions are continuous.

Recall that continuity at a means that

$$\lim_{x \to a} f(x) = f(a)$$

and this is **direct substitution**.

Evaluate $\lim_{x \to \pi} f(x)$ where $f(x) = \frac{\sin x}{2 + \cos x}$.

We know that sin, cos and 2 are continuous functions.
Then their sum and quotient are continuous on their domain.
The domain contains π, so: $\lim_{x \to \pi} f(x) = f(\pi) = 0/(2 - 1) = 0$.

Continuity: Function Composition

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a}(g(x))$$

Evaluate $\lim_{x\to 4} \sin(\frac{\pi}{4+\sqrt{x}})$. We have

$$\begin{aligned}
\lim_{x\to 4} \sin(\frac{\pi}{4+\sqrt{x}}) &= \sin(\lim_{x\to 4} \frac{\pi}{4+\sqrt{x}}) && \text{since sin is continuous} \\
&= \sin(\frac{\pi}{4+\sqrt{4}}) && \text{direct substitution} \\
&= \sin(\frac{\pi}{6}) = \frac{1}{2}
\end{aligned}$$

Continuity: Function Composition

The composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

If

- g is continuous at a, and
- f is continuous at $g(a)$,

then the composite function $f \circ g$ is continuous at a.

A continuous function of a continuous function is continuous.

Where is $h(x) = \sin x^2$ continuous?
Both x^2 and sin are continuous everywhere (on $(-\infty, \infty)$).
Thus $h(x)$ is continuous everywhere.

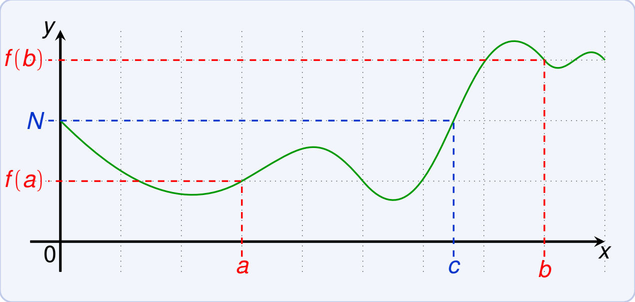
Where is $h(x) = \ln(1 + \cos x)$ continuous?
The functions 1, cos (and their sum) and ln are on their domain.
Thus $h(x)$ is continuous on its domain: $\mathbb{R} \setminus \{\pm\pi, \pm 3\pi, \pm 5\pi, \ldots\}$.

Continuity: Intermediate Value Theorem

Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. If N is strictly between $f(a)$ and $f(b)$. Then

$$f(c) = N \qquad \text{for some number } c \text{ in } (a, b)$$



Every N between $f(a)$ and $f(b)$ occurs at least once on (a, b).
Intuitively: the graph cannot jump over the line $y = N$.

Continuity: Intermediate Value Theorem

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

We are looking for number c such that $f(c) = 0$ and $1 < c < 2$.

We have:

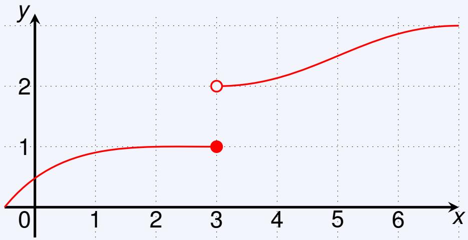
- the function is continuous on the interval since it is a polynomial
- $f(1) = 4 - 6 + 3 - 2 = -1$
- $f(2) = 4 \cdot 8 - 6 \cdot 4 + 3 \cdot 2 - 2 = 12$

Moreover $-1 < 0 < 12$. Thus we can apply the Intermediate Value Theorem for the interval $[1, 2]$ and $N = 0$.

Hence there exists c in $(1, 2)$ such that $f(c) = 0$.

Continuity: Intermediate Value Theorem

Whenever applying the Intermediate Value Theorem, it is **important** to check that the function is **continuous** on the interval.



Here we have:

- $f(2) < 1$
- $f(4) > 2$

But there exists no $2 < c < 4$ such that $f(c) = 1.5$!

Continuity: Intermediate Value Theorem

Show that the following equation

$$6 \cdot 3^{-x} = 4 - x$$

has a solution for x in $[0, 1]$.

Define

$$6 \cdot 3^{-x} = 4 - x \quad \Longleftrightarrow \quad 6 \cdot 3^{-x} + x - 4 = 0$$

The function $f(x) = 6 \cdot 3^{-x} + x - 4$ is a sum and product of continuous functions, and hence continuous.

We have:

- $f(0) = 6 \cdot 3^0 + 0 - 4 = 2$
- $f(1) = 6 \cdot 3^{-1} + 1 - 4 = -1$

Moreover $-1 < 0 < 2$.

By the Intermediate Value Theorem there exists x in the interval $[0, 1]$ such that $f(x) = 0$. This x is a solution of the equation.