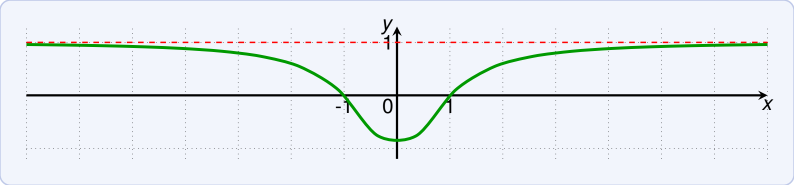
Limits at Infinity

Lets investigate the behavior of the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

when x becomes large:



As x grows larger, the values of $f(x)$ get closer and closer to 1.

This is expressed by

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

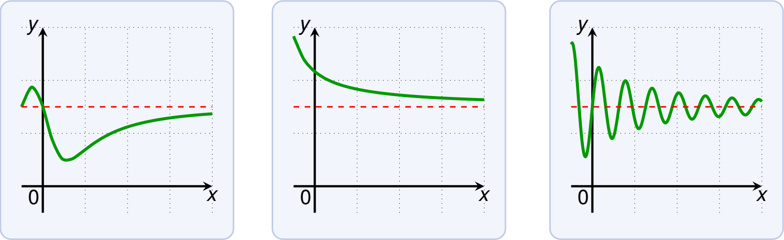
Limits at Infinity

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches infinity, is L"

if the values $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



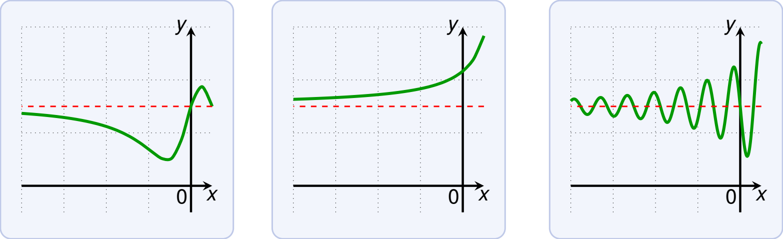
Limits at Infinity

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches negative infinity, is L"

if the values $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

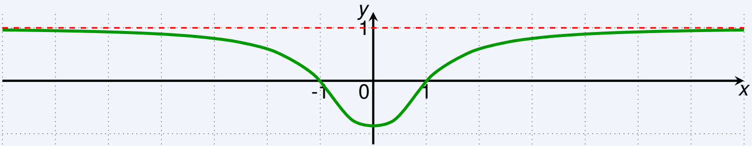


Limits at Infinity: Horizontal Asymptotes

The line $y = L$ is called **horizontal asymptote** of a function f if

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

The function $f(x) = \frac{x^2-1}{x^2+1}$ has a horizontal asymptote at $y = 1$.



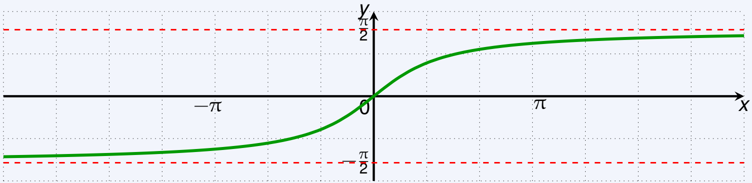
Limits at Infinity: Horizontal Asymptotes

The line $y = L$ is called **horizontal asymptote** of a function f if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

The inverse tangent $\tan^{-1}$ has horizontal asymptotes

$$y = -\frac{\pi}{2} \qquad \text{and} \qquad y = \frac{\pi}{2}$$



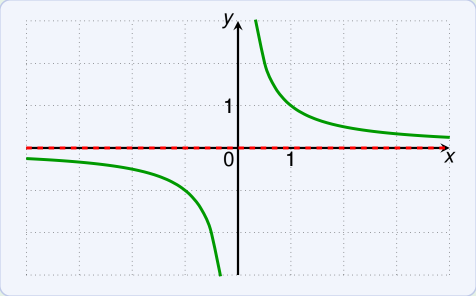
$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

Limits at Infinity

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

As x gets larger, $\frac{1}{x}$ gets closer to 0.
Thus $\lim_{x \to \infty} \frac{1}{x} = 0$.

As x gets larger negative, $\frac{1}{x}$ gets closer to 0.
Thus $\lim_{x \to -\infty} \frac{1}{x} = 0$.



The function has the horizontal asymptote $y = 0$.

Limits at Infinity: Laws

All **limits laws** for $\lim_{x\to a}$ work also for $\lim_{x\to\pm\infty}$, except for:

$$\lim_{x\to a} x^n = a^n \qquad\qquad \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

For example, we can derive the following important theorem:

For $r > 0$ we have

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

and if x^r is defined for all x, then also

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Proof

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to\infty} (\frac{1}{x})^r = (\lim_{x\to\infty} \frac{1}{x})^r = 0^r = 0$$

Limits at Infinity

Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x\to\infty} \left(\frac{3x^2 - x - 2}{5x^2 + 4x + 1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \right) \\
&= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\
&= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \\
&= \frac{3}{5}
\end{aligned}$$

Limits at Infinity

Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

We have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \left(\frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{(\frac{1}{x})}{(\frac{1}{x})} \right)$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3-\frac{5}{x}} \quad \text{since } x > 0,\ x = \sqrt{x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\lim_{x\to\infty} \sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}(3-\frac{5}{x})}$$

$$= \frac{\sqrt{\lim_{x\to\infty}(2+\frac{1}{x^2})}}{3} = \frac{\sqrt{2}}{3}$$

Limits at Infinity

Evaluate

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

We have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \left(\frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \right)$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3-\frac{5}{x}} \quad \text{since } x < 0,\ x = -\sqrt{x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)}}{3} = \frac{\sqrt{2}}{3}$$

Limits at Infinity

Evaluate

$$\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

We have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty}\left(\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{(\frac{1}{x})}{(\frac{1}{x})}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{3-\frac{5}{x}} \quad \text{since } x<0,\ x=-\sqrt{x^2}$$

$$= \lim_{x\to\infty} - \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = - \frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}(3-\frac{5}{x})}$$

$$= - \frac{\sqrt{\lim_{x\to\infty}(2+\frac{1}{x^2})}}{3} = - \frac{\sqrt{2}}{3}$$

Limits at Infinity

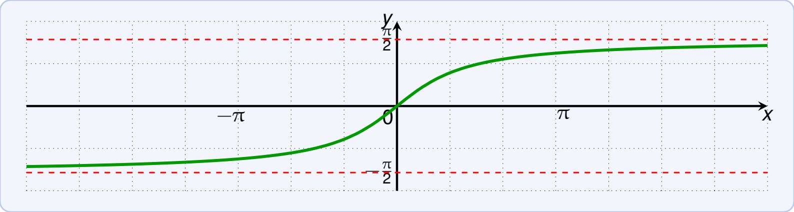
Evaluate

$$\lim_{x\to\infty} (\sqrt{x^2-1} - x)$$

We have

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2-1} - x) &= \lim_{x\to\infty} \left(\frac{\sqrt{x^2-1} - x}{1} \cdot \frac{\sqrt{x^2-1} + x}{\sqrt{x^2-1} + x} \right) \\
&= \lim_{x\to\infty} \frac{x^2 - 1 - x^2}{\sqrt{x^2-1} + x} \\
&= \lim_{x\to\infty} -\frac{1}{\sqrt{x^2-1} + x} \\
&= \lim_{x\to\infty} -\frac{1}{\sqrt{x^2-1} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} -\frac{\frac{1}{x}}{\sqrt{1 - \frac{1}{x^2}} + 1} = \frac{0}{2} = 0
\end{aligned}$$

Limits at Infinity



The graph of $\tan^{-1}$.

Evaluate

$$\lim_{x \to 2+} \tan^{-1}\left(\frac{1}{x-2}\right) = \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

Limits at Infinity

For exponential function we have:

$$\lim_{x \to \infty} a^x = 0 \qquad \text{for } 0 \leq a < 1$$

$$\lim_{x \to -\infty} a^x = 0 \qquad \text{for } a > 1$$

For any polynomial P and $a > 1$ we have

$$\lim_{x \to \infty} \frac{P(x)}{a^x} = 0$$

since the exponential function grows after than any polynomial.

For any polynomial P and $0 < a < 1$ we have

$$\lim_{x \to -\infty} \frac{P(x)}{a^x} = 0$$

Limits at Infinity

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

A good heuristic (this is not a law) for to look at:

- the fastest growing addend of $f(x)$
- the fastest growing addend of $g(x)$

Typically, the other addends do not matter.

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

$$\lim_{x \to \infty} \frac{\sqrt{5x^3 + 1} + 2x^2}{x^2 + 1} = 2$$

$$\lim_{x \to \infty} \frac{5x^3 + x + x \cdot x^2}{2x^3 - x} = 3$$

Limits at Infinity

Evaluate

$$\lim_{x\to\infty} \frac{3x^5 + x^2 - 2}{x^2 - x + 2^x} = 0$$

since 2^x grows faster than any polynomial.

Evaluate

$$\lim_{x\to\infty} \frac{3x^2 + x}{5x^2 - x + 5^{-x}} = \frac{3}{5}$$

since $\lim_{x\to\infty} 5^{-x} = 0$.

Evaluate

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{x\to -\infty} e^x = 0$$

Limits at Infinity

Evaluate

$$\lim_{x\to\infty} \sin(x) = \text{does not exist}$$

since $\sin(x)$ oscillates between -1 and 1.

Evaluate

$$\lim_{x\to\infty} \frac{3\sin(x)}{x^2} = 0$$

since the denominator grows to infinity while $-3 \leq 3\sin(x) \leq 3$.

Evaluate

$$\lim_{x\to\infty} \frac{2x^3 + x^2 \cdot \cos(x) + 3e^x + x}{x^5 + 5e^x} = \frac{3}{5}$$

since the exponential functions grow much faster than the rest. To use limit laws, multiply numerator and denominator by $\frac{1}{e^x}$.

Infinite Limits at Infinity

$$\lim_{x \to \infty} f(x) = \infty$$

if we can make the values of $f(x)$ arbitrary large by taking x sufficiently large.

Similar for:

$$\lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to \infty} x^3 = \infty \qquad \lim_{x \to -\infty} x^3 = -\infty$$

$$\lim_{x \to \infty} a^x = \infty \qquad \text{for } a > 1$$

$$\lim_{x \to -\infty} a^x = \infty \qquad \text{for } 0 < a < 1$$

Infinite Limits at Infinity

Evaluate

$$\lim_{x \to \infty} (x^2 - x)$$

The limit laws do not help since:

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty = \text{invalid expression}$$

However, we can write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large.

Infinite Limits at Infinity: Heuristics

All on this slide is heuristics, not laws!

On the last slide we could have reasoned as follows:

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x \cdot \lim_{x\to\infty}(x-1) = \infty \cdot \infty = \infty$$

Valid calculations with ∞ and x a real number:

$$\infty + \infty = \infty \qquad \infty + x = \infty \qquad \frac{x}{\infty} = 0$$

$$\frac{\infty}{x} = \infty \text{ if } x > 0 \qquad \frac{\infty}{x} = -\infty \text{ if } x < 0$$

Invalid, undefined expressions:

$$\infty - \infty \qquad \infty + (-\infty) \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty$$

Infinite Limits at Infinity

Evaluate

$$\lim_{x\to\infty} \frac{x^2+x}{3-x}$$

We have

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2+x}{3-x} &= \lim_{x\to\infty} \left(\frac{x^2+x}{3-x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \right) \\ &= \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1} \\ &= \frac{\infty}{0-1} \\ &= -\infty\end{aligned}$$

because $x+1$ grows to infinity while $\frac{3}{x}-1$ gets closer to -1.