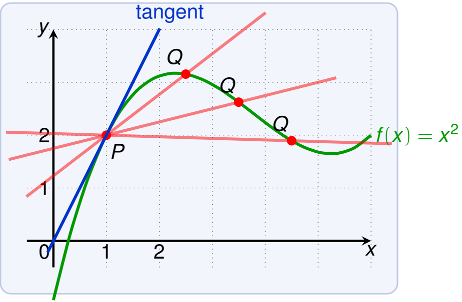
Finding a Tangent

We move Q closer and closer to P.



The limit is the tangent.

The **tangent line** to the curve $f(x)$ at point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

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Find an equation of the tangent line to $f(x) = x^2$ at point $(1, 1)$.

We use the equation for the slope with $a = 1$:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

Thus $y - 1 = 2(x - 1)$, that is, the tangent is $y = 2x - 1$.

Finding a Tangent

Alternative definition of the slope:

The slope of f at point $(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The slope m is also called the **slope of the curve** at the point.

Find an equation of the tangent to $f(x) = \frac{3}{x}$ at point $(3, 1)$.

The slope is:

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Thus $y - 1 = -\frac{1}{3}(x - 3)$, that is, the tangent is $y = 2 - \frac{x}{3}$.

Velocities

Let $f(t)$ be a **position function** of an object:

- $f(t)$ is the position (distance form the origin) after time t

The average velocity in the time interval $(a, a+h)$ is

$$\text{average velocity} = \frac{\text{difference in position}}{\text{time difference}} = \frac{f(a+h) - f(a)}{h}$$

The (instantaneous) **velocity** $v(a)$ at time $t = a$ is:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

which is the slope of the tangent at point $(a, f(a))$.

Let $f(t) = 2t^2$. What is the speed of the object after n seconds?

$$v(n) = \lim_{h \to 0} \frac{2 \cdot (n+h)^2 - 2 \cdot n^2}{h} = \lim_{h \to 0} \frac{4nh + 2 \cdot h^2}{h}$$

$$= \lim_{h \to 0} (4n + 2 \cdot h) = 4n$$

Derivatives

The **derivative of a function** f **at a number** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exits.

Find the derivative of $f(x) = x^2 - 8x + 9$ at number a.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

Derivatives

The **derivative of a function** f **at a number** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exits.

An equivalent way of defining the derivative (take $x = a + h$):

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

The tangent line to f at point $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$, the derivative of f at a.

Find an equation of the tangent to $f(x) = x^2 - 8x + 9$ at $(3, -6)$.

We know $f'(a) = 2a - 8$, and thus $f'(3) = -2$.

Hence $y + 6 = -2(x - 3)$,that is, $y = -2x$

Rates of Change

Suppose y is a quantity that depends on x. That is $y = f(x)$.

If x changes from x_1 to x_2, the change (increment) of x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The **average rate of change over the interval** $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The **instantaneous rate of change** by letting Δx go to 0:

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This is the derivative $f'(x_1)$!

(Note that large derivative $f'(x_1)$ means rapid change.)

Rates of Change

A manufacturer produces some fabric. The costs for producing x yards are $f(x)$ dollars.

- What is the meaning of $f'(x)$ (called **marginal costs**)?
- What does it mean to say $f'(1000) = 9$?
- Which do you think is greater $f'(50)$ or $f'(500)$?

Answers:

- $f'(x)$ is the rate of change of production costs in dollars per yard with respect to the number of yards produced
- $f'(1000) = 9$ means that after having produced 1000 yards, the costs increase by 9 dollars for additional yards
- Typically $f'(500) < f'(50)$ since usually the cost of production per yard will decrease the more you produce (due to fixed costs: you have already bought and installed the machines...)

Rates of Change

Rates of change are important in:

- all natural sciences,
- in engineering, and
- social sciences

Examples of rate of change:

- in economics: change of production costs with respect to the number of items produced (called marginal costs)
- in physics: rate of change of work with respect to time (called power)
- in chemistry: rate of change of the concentration of a reactant with respect to time (called rate of reaction)
- in biology: rate of change of the population of bacteria with respect to time