# Derivatives of Basic Functions

The derivative of a constant function

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

If $n$ is any real number, then

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

Differentiate the following functions:

- $\frac{d}{dx}(x^7) = 7x^6$
- $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2\,x^{-3} = -\frac{2}{x^3}$
- $\frac{d}{dx}(\sqrt[3]{x^2}) = \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}\,x^{\frac{2}{3}-1} = \frac{2}{3}\,x^{-\frac{1}{3}}$

# Derivatives of Basic Functions

The **normal line** is perpendicular to the tangent.

If the tangent has slope $m$, then the normal line has slope $-\frac{1}{m}$.

Find equations for the tangent and normal line to $x\sqrt{x}$ at $(1, 1)$.

$$f'(x) = \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{1.5}) = 1.5x^{.5} = \frac{3}{2}\sqrt{x}$$

The slope of the tangent at $(1, 1)$ is $\frac{3}{2}$. Hence the tangent is

$$y - 1 = \frac{3}{2}(x - 1) \qquad\qquad y = \frac{3}{2}x - \frac{1}{2}$$

The slope of the normal at $(1, 1)$ is $-1/\frac{3}{2} = -\frac{2}{3}$. Hence the normal is

$$y - 1 = -\frac{2}{3}(x - 1) \qquad\qquad y = -\frac{2}{3}x + \frac{5}{3}$$

# Derivatives of Basic Functions

## Constant Multiple Rule

If $c$ is a constant and $f$ is differentiable, then

$$\frac{d}{dx}[c\,f(x)] \;=\; c \cdot \frac{d}{dx}f(x)$$

## Sum Rule

If $f$ and $g$ are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] \;=\; \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

## Difference Rule

If $f$ and $g$ are differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] \;=\; \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

# Derivatives of Basic Functions

Compute the following derivative:

$$\begin{aligned}
&\frac{d}{dx}(12x^5 - 10x^3 - 6x + 5) \\
&\quad = 12\frac{d}{dx}(x^5) - 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5) \\
&\quad = 12 \cdot 5x^4 - 10 \cdot 3x^2 - 6 \cdot 1 + 0 = 60x^4 - 30x^2 - 6
\end{aligned}$$

The motion of a particle is given by:

- $s(t) = 2t^3 - 5t^2 + 3t + 4$ ($t$ is in seconds, and $s(t)$ in cm)

Find the acceleration function, and the acceleration after $2s$.

$$v(t) = \frac{d}{dt}s(t) = 6t^2 - 10t + 3 \qquad \text{in cm/s}$$

$$a(t) = \frac{d}{dt}v(t) = 12t - 10 \qquad \text{in cm/s}^2$$

The acceleration after $2s$ is $14\text{cm}/\text{s}^2$.

# Derivatives of Basic Functions

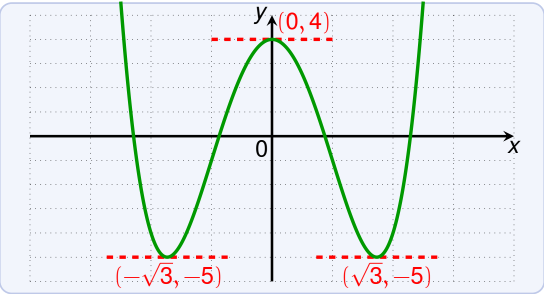
Find the points of $f(x) = x^4 - 6x^2 + 4$ with horizontal tangent.

Horizontal tangent means that the slope (the derivative) is 0:

$$\frac{d}{dx} f(x) \;=\; 4x^3 - 12x \;=\; 4x(x^2 - 3)$$

Thus $f'(x) = 0$ when $x = 0$ or $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Thus the corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, $(-\sqrt{3}, -5)$.



# Derivatives of Basic Functions

## Sum Rule

If $f$ and $g$ are differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

## Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x)+\frac{d}{dx}g(x)
\end{aligned}$$

□

# Derivatives of Exponential Functions

We compute the derivative of $f(x) = a^x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h - 1}{h} = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$$

Note that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)$$

For $f(x) = a^x$ we have

$$f'(x) = f'(0) \cdot a^x$$

Note that slope is proportional to the function itself.

# Derivatives of Exponential Functions

For $f(x) = a^x$ we have

$$f'(x) = f'(0) \cdot a^x$$

Using the calculator we can estimate that:

for $a = 2$ $\quad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$

for $a = 3$ $\quad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$

There is a number $a$ between 2 and 3 such that $f'(0) = 1$:

$e$ is the number such that $\quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$

The function $e^x$ is the only exponential with slope 1 at $(0, 1)$.

# Derivatives of Exponential Functions

$$\frac{d}{dx}(e^x) \ = \ e^x$$

$$\frac{d}{dx}(a^x) \ = \ \ln a \cdot a^x$$

At what point on the curve $e^x$ is the tangent parallel to $y = 2x$?

Let $f(x) = e^x$

$$f'(a) = e^a = 2$$

Thus $a = \ln 2$, that is, the point is $(a, e^a) = (\ln 2, 2)$.

# Derivatives of Exponential Functions

Let $f(x) = e^x - x$. Find $f'$ and $f''$.

$$f'(x) = e^x - 1 \qquad\qquad f''(x) = e^x$$