# Derivative as a Function

The **derivative of** $f$ is a function $f'$ defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- The domain of $f'$ is the set $\{x \mid f'(x) \text{ exists}\}$.
- Geometrically, $f'(x)$ is the slope of the tangent at $(x, f(x))$.

Let $f(x) = x^3 - x$. Find a formula for $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) \\
&= 3x^2 - 1
\end{aligned}$$

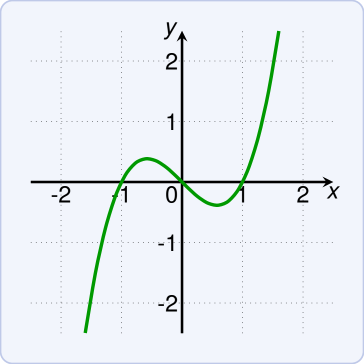
# Exam Task from 2005

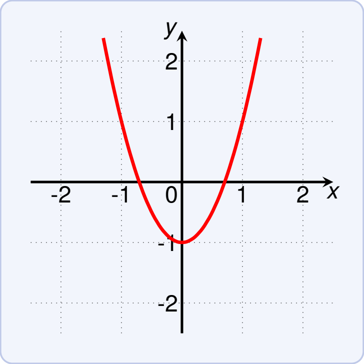
Using the definition of derivative, find $f'(x)$, where $f(x) = \sqrt{2x}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \\
&= \lim_{h\to 0} \left( \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \cdot \frac{\sqrt{2x+2h} + \sqrt{2x}}{\sqrt{2x+2h} + \sqrt{2x}} \right) \\
&= \lim_{h\to 0} \left( \frac{2x + 2h - 2x}{h \cdot (\ \sqrt{2x+2h} + \sqrt{2x}\ )} \right) \\
&= \lim_{h\to 0} \left( \frac{2}{\sqrt{2x+2h} + \sqrt{2x}} \right) \\
&= \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}
\end{aligned}
$$

# Derivative as a Function

Which of these functions is the derivative of the other?




The right is the derivative of the left:

- look at local maxima and minima of $f$; then $f'$ must be 0
- where $f$ increases, $f'$ must be positive
- where $f$ decreases, $f'$ must be negative

# Derivative as a Function

A function $f$ is **differentiable at** $a$ if $f'(a)$ exists.

A function $f$ is **differentiable on an open interval** $(a, b)$ if it is differentiable at every number of the interval.

Note that the interval $(a, b)$ may be $(-\infty, b)$, $(a, \infty)$ or $(-\infty, \infty)$.

# Derivative as a Function

Where is $f(x) = |x|$ differentiable?

For $x > 0$ we have:

- $|x| = x$,
- $|x+h| = x+h$ for small enough $h$.

Thus for $x > 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} 1 = 1$$

For $x < 0$ we have:

- $|x| = -x$,
- $|x+h| = -x-h$ for small enough $h$.

Thus for $x < 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-x-h+x}{h} = \lim_{h\to 0} -1 = -1$$

# Derivative as a Function

Where is $f(x) = |x|$ differentiable?

For $x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

We need to look at the left and right limits:

$$\lim_{h \to 0^-} \frac{|h|}{h} \overset{\text{since } h<0}{=} \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

and

$$\lim_{h \to 0^+} \frac{|h|}{h} \overset{\text{since } h>0}{=} \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

The left and right limits are different.

Thus $f'(0)$ does not exist, and $f(x)$ is not differentiable at 0.

Hence $f$ is differentiable at all numbers in $(-\infty, 0) \cup (0, \infty)$.

# Derivatives and Continuity

If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

The proof is in the book. Intuitively it holds because...

Differentiable at $a$ means:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists}$$

Continuous at $a$ means:

$$\lim_{x \to a} f(x) = f(a) \quad \iff \quad \lim_{x \to a} (f(x) - f(a)) = 0$$

$$\iff \quad \lim_{h \to 0} (f(a+h) - f(a)) = 0$$

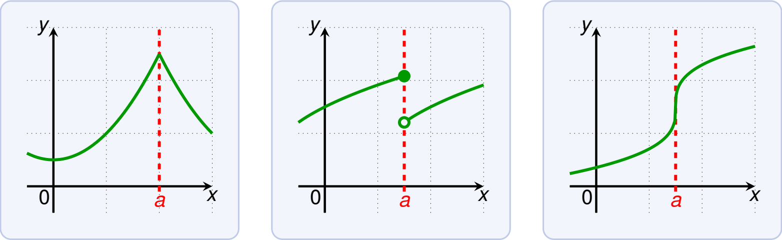
If the latter limit would not be 0 (or not exist),
then $\frac{f(a+h)-f(a)}{h}$ would get arbitrarily large for small $h$.

If $f$ is continuous at $a$, then $f$ is not always differentiable at $a$.

E.g. $|x|$ is continuous at 0 but not differentiable at 0.

# How can a Function fail to be Derivable?

There are the following reasons for failure of being derivable:




- graph changes direction abruptly (graph has a "corner")
- the function is not continuous at $a$
- graph has a vertical tangent at $a$, that is:

$$\lim_{x \to a} |f'(x)| = \infty$$

Example for a vertical tangent is $f(x) = \sqrt[3]{x}$ at 0.

# Derivative: Other Notations

We usually write $f'(x)$ for the derivative.

However, there are other common notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols $\frac{d}{dx}$ and $D$ are called **differentiation operators.**
(they indicate the operation of computing the derivative)

The notation $\frac{dy}{dx}$ has been introduced by Leibnitz:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

In Leibnitz notation $f'(a)$ is written as

$$\left.\frac{dy}{dx}\right|_a \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_a$$

# Higher Derivatives

If $f$ is a function, the derivative $f'$ is also a function.

Thus we can compute the derivative of the derivative:

$$(f')' = f''$$

The function $f''$ is called **second derivative** of $f$.

Let $f(x) = x^3 - x$. Find $f''(x)$.

We have seen $f'(x) = 3x^2 - 1$. Thus

$$\begin{aligned}
f''(x) = (f')'(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} (6x + 3h) = 6x
\end{aligned}$$

# Higher Derivatives

What is the meaning of $f''(x)$?

- the slope of $f'(x)$ at point $(x, f'(x))$
- the rate of change of $f'(x)$
- the rate of change of the rate of change of $f(x)$

The **acceleration** is an example of a second derivative:

- $s(t)$ is the position of an object (at time $t$)
- $v(t) = s'(t)$ is the speed (at time $t$)
- $a(t) = v'(t) = s''(t)$ is the acceleration (at time $t$)

# Higher Derivatives

We can continue this process of deriving:

- $f'''(x) = (f'')'(x)$
- $f''''(x) = (f''')'(x)$
- $\ldots$

The *n*-th derivative of $f$ is denoted by

$$f^{(n)}(x) \qquad \text{or} \qquad \frac{d^n y}{dx^n}$$

For example, $f = f^{(0)}, \quad f' = f^{(1)}, \quad f'' = f^{(2)}, \quad f''' = f^{(3)}$

Let $f(x) = x^3 - x$. Find $f'''(x)$ and $f^{(4)}(x)$.

We know $f''(x) = 6x$. Hence

$$f'''(x) = 6 \qquad f^{(4)}(x) = 0$$

Note that $f'''$ is the slope of $f''$, and $f^{(4)}$ is the slope of $f'''$.