Differentiation Rules: Product Rule

Lets f and g be linear functions:

$$f(x) = ax + b \qquad\qquad g(x) = cx + d$$

What is the derivative of $f \cdot g$?

$$\begin{aligned}
(f \cdot g)'(x) &= \frac{d}{dx}[f(x) \cdot g(x)] \\
&= \frac{d}{dx}[(ax + b) \cdot (cx + d)] \\
&= \frac{d}{dx}[acx^2 + adx + bcx + bd] \\
&= 2acx + ad + bc \\
&= a(cx + d) + c(ax + b) \\
&= f'(x) \cdot g(x) + g'(x) \cdot f(x)
\end{aligned}$$

We will now see that this also holds for general f and g.

Differentiation Rules: Product Rule

Assume that f and g are differentiable at x, and define

$$h(x) = f(x) \cdot g(x)$$

We try to find the derivative of h at x:

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \text{where} \qquad \begin{aligned} \Delta h &= h(x + \Delta x) - h(x) \\ \Delta f &= f(x + \Delta x) - f(x) \\ \Delta g &= g(x + \Delta x) - g(x) \end{aligned}$$

Then

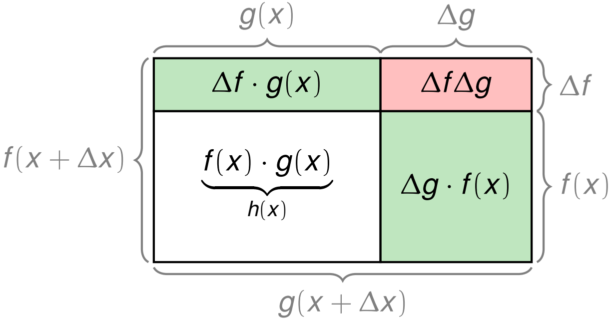
$$\begin{aligned} \Delta h &= f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x) \\ &= (f(x) + \Delta f) \cdot (g(x) + \Delta g) - f(x) \cdot g(x) \\ &= \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g \end{aligned}$$

Differentiation Rules: Product Rule

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \text{where} \qquad \begin{aligned} \Delta h &= h(x + \Delta x) - h(x) \\ \Delta f &= f(x + \Delta x) - f(x) \\ \Delta g &= g(x + \Delta x) - g(x) \end{aligned}$$

Then

$$\Delta h = \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g$$



Differentiation Rules: Product Rule

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \Delta h \;= \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g$$

We compute the limit:

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta f \cdot g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x) \cdot \Delta g}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta f \cdot \Delta g}{\Delta x} \\
&= g(x) \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + f(x) \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} + \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x} \cdot \Delta g \right) \\
&= g(x) f'(x) + f(x) g'(x) + \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta g \\
&= g(x) f'(x) + f(x) g'(x) + f'(x) \cdot 0 \\
&= g(x) f'(x) + f(x) g'(x)
\end{aligned}$$

Differentiation Rules: Product Rule

Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$$

In different notation

$$(f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

In words:

> The derivative of the product of two function is the first function times the derivative of the second function plus the second function times the derivative of the first.

Differentiation Rules: Product Rule

Let $f(x) = xe^x$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x \cdot e^x) \\ &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\ &= xe^x + e^x \\ &= (x+1)e^x \end{aligned}$$

Let $f(x) = xe^x$. Find the n-th derivative $f^{(n)}(x)$.

$$f''(x) = \frac{d}{dx}(xe^x + e^x) = (x+1)e^x + e^x = (x+2)e^x$$

$$f'''(x) = \frac{d}{dx}(xe^x + 2e^x) = (x+1)e^x + 2e^x = (x+3)e^x$$

Thus obviously we have

$$f^{(n)}(x) = (x+n)e^x$$

Differentiation Rules: Product Rule

Differentiate $f(t) = \sqrt{t}(a + bt)$.

$$
\begin{aligned}
f'(t) &= \sqrt{t}\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}(\sqrt{t}) \\
&= \sqrt{t}b + (a + bt)\frac{1}{2}t^{-\frac{1}{2}}) \\
&= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} \\
&= \frac{2bt}{2\sqrt{t}} + \frac{a + bt}{2\sqrt{t}} \\
&= \frac{a + 3bt}{2\sqrt{t}}
\end{aligned}
$$

Alternative solution: first simplify $f(t) = \sqrt{t}(a + bt) = at^{\frac{1}{2}} + bt^{\frac{3}{2}}$. Then compute the derivative.

Differentiation Rules: Product Rule

Let $f(x) = \sqrt{x} \cdot g(x)$ where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

$$
\begin{aligned}
f'(x) &= \sqrt{x} \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} \sqrt{x} \\
&= \sqrt{x} \cdot g'(x) + g(x) \cdot \frac{1}{2} x^{-\frac{1}{2}} \\
&= \sqrt{x} \cdot g'(x) + g(x) \cdot \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
f'(4) &= \sqrt{4} \cdot g'(4) + g(4) \cdot \frac{1}{2\sqrt{4}} \\
&= 2 \cdot 3 + 2 \cdot \frac{1}{2 \cdot 2} \\
&= 6 + \frac{1}{2} \\
&= \frac{13}{2}
\end{aligned}
$$

Differentiation Rules: Quotient Rule

Assume that f and g are differentiable at x, and define

$$h(x) = \frac{f(x)}{g(x)} \qquad \begin{aligned} \Delta h &= h(x + \Delta x) - h(x) \\ \Delta f &= f(x + \Delta x) - f(x) \\ \Delta g &= g(x + \Delta x) - g(x) \end{aligned}$$

We try to find the derivative of h at x:

$$\begin{aligned} \Delta h = h(x + \Delta x) - h(x) &= \frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)} = \frac{f(x) + \Delta f}{g(x) + \Delta g} - \frac{f(x)}{g(x)} \\ &= \frac{(f(x) + \Delta f) \cdot g(x) - (g(x) + \Delta g) \cdot f(x)}{(g(x) + \Delta g) \cdot g(x)} = \frac{g(x)\Delta f - f(x)\Delta g}{(g(x) + \Delta g) \cdot g(x)} \end{aligned}$$

$$\begin{aligned} {\color{red} h'(x)} &= \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{g(x)\Delta f - f(x)\Delta g}{(g(x)+\Delta g)\cdot g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x)\frac{\Delta f}{\Delta x} - f(x)\frac{\Delta g}{\Delta x}}{(g(x) + \Delta g) \cdot g(x)} \\ &= \frac{g(x)\lim_{\Delta x \to 0}\frac{\Delta f}{\Delta x} - f(x)\lim_{\Delta x \to 0}\frac{\Delta g}{\Delta x}}{\lim_{\Delta x \to 0}(g(x) + \Delta g) \cdot g(x)} = {\color{red} \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}} \end{aligned}$$

Differentiation Rules: Quotient Rule

Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}(f(x)) - f(x)\cdot\frac{d}{dx}(g(x))}{[g(x)]^2}$$

In different notation

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

In words:

> The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Differentiation Rules: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Let

$$f(x) = \frac{x^2 + x - 2}{x^3 + 6}$$

Then

$$\begin{aligned}
f'(x) &= \frac{(x^3+6)\cdot \frac{d}{dx}(x^2+x-2) - (x^2+x-2)\cdot \frac{d}{dx}(x^3+6)}{(x^3+6)^2} \\
&= \frac{(x^3+6)\cdot(2x+1) - (x^2+x-2)\cdot 3x^2}{(x^3+6)^2} \\
&= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2} \\
&= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}
\end{aligned}$$

Differentiation Rules: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) \; = \; \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Find an equation to the tangent line to

$$f(x) = \frac{e^x}{1+x^2}$$

at point $(1, \frac{e}{2})$. We have

$$f'(x) = \frac{(1+x^2)\cdot \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(1+x^2)}{(1+x^2)^2} = \frac{(1+x^2)e^x - e^x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{x^2e^x - 2xe^x + e^x}{(1+x^2)^2} = \frac{(x-1)^2e^x}{(1+x^2)^2}$$

Thus the slope of the tangent is $f'(1) = 0$. Hence the tangent is

$$y = \frac{e}{2}$$

Differentiation Rules: Quotient Rule

Sometimes it is easier to simplify than apply the quotient rule:

$$f(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

Instead of applying the quotient rule, we can simplify to

$$f(x) = 3x + 2x^{-\frac{1}{2}}$$

which is easier to differentiate.

Differentiation Rules: Chain Rule

Suppose we want to differentiate

$$f(x) = \sqrt{x^2 + 1}$$

The rules, we have seen so far, do not help.

However, we know how to differentiate the functions:

$$g(x) = \sqrt{x} \qquad\qquad h(x) = x^2 + 1$$

We can write f as:

$$f(x) = g(h(x))$$

That is:

$$f = g \circ h$$

We need a rule that gives us f' from g' and h'...

Differentiation Rules: Chain Rule

Chain Rule

If g is differentiable at x and f at $g(x)$, then

$$h = f \circ g \qquad \text{or equivalently} \qquad h(x) = f(g(x))$$

is differentiable at x and

$$h'(x) \;=\; (f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} \underbrace{f}_{\substack{\text{outer}\\ \text{function}}} \underbrace{(\;g(x)\;)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \underbrace{(\;g(x)\;)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

In words:

The derivative of the composition of f and g is the derivative of f at $g(x)$ times the derivative of g at x.

Differentiation Rules: Chain Rule

Chain Rule

If g is differentiable at x and f at $g(x)$, then

$$h = f \circ g \qquad \text{or equivalently} \qquad h(x) = f(g(x))$$

is differentiable at x and

$$h'(x) \;=\; (f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Intuition with rates of change:

- If $g'(x) = N$. Then $g(x)$ changes N times as much as x.
- If $f'(g(x)) = M$. Then $f(x)$ changes M times as much as $g(x)$.
- Thus $(f \circ g)(x) = f(g(x))$ changes $N \cdot M$ times as much as x.

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Let $f(x) = \sqrt{x^2 + 1}$. Find $f'(x)$.

We have that

$$f(x) = g(h(x)) \qquad \text{where} \qquad g(x) = \sqrt{x} \qquad h(x) = x^2 + 1$$

and

$$g'(x) = \frac{1}{2\sqrt{x}} \qquad\qquad h'(x) = 2x$$

Hence:

$$f'(x) = (g \circ h)'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Differentiate $f(x) = (x^3 - 1)^{100}$.

We have that

$$f(x) = g(h(x)) \qquad \text{where} \qquad g(x) = x^{100} \qquad h(x) = x^3 - 1$$

and

$$g'(x) = 100x^{99} \qquad\qquad h'(x) = 3x^2$$

Hence:

$$\begin{aligned} f'(x) = (g \circ h)'(x) &= 100(x^3 - 1)^{99} \cdot 3x^2 \\ &= 300x^2 \cdot (x^3 - 1)^{99} \end{aligned}$$

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

In general (combining the power and chain rule) we have:

$$\frac{d}{dx}[g(x)]^n \;=\; n \cdot [g(x)]^{n-1} \cdot g'(x)$$

if $g(x)$ is differentiable.

Differentiate

$$f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$$

We have

$$f(x) = (x^2 + x + 1)^{-\frac{1}{3}}$$
$$f'(x) = -\frac{1}{3} \cdot (x^2 + x + 1)^{-\frac{4}{3}} \cdot (2x + 1)$$

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Differentiate

$$f(x) = \left(\frac{x-2}{2x+1}\right)^9$$

We have

$$\begin{aligned} f'(x) &= 9\left(\frac{x-2}{2x+1}\right)^8 \frac{d}{dx}\frac{x-2}{2x+1} \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \frac{(2x+1)\cdot 1-(x-2)\cdot 2}{(2x+1)^2} \\ &= 9\left(\frac{x-2}{2x+1}\right)^8 \frac{5}{(2x+1)^2} \\ &= 45\frac{(x-2)^8}{(2x+1)^{10}} \end{aligned}$$

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Differentiate

$$f(x) = (2x+1)^5 \cdot (x^3 - x + 1)^4$$

We have

$$\begin{aligned} f'(x) &= (2x+1)^5 \cdot \frac{d}{dx}[(x^3 - x + 1)^4] \\ &\quad + (x^3 - x + 1)^4 \cdot \frac{d}{dx}[(2x+1)^5] \\ &= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \cdot (3x^2 - 1) \\ &\quad + (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \cdot 2 \end{aligned}$$

Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Use

$$\frac{d}{dx} e^x = e^x$$

and the chain rule to prove

$$\frac{d}{dx} a^x = \ln a \cdot a^x$$

We have

$$a^x \;=\; (e^{\ln a})^x = e^{\ln a \cdot x}$$

and $f = g \circ h$ where $g(x) = e^x$ and $h(x) = \ln a \cdot x$. Thus

$$f'(x) = g'(h(x)) \cdot h'(x) = e^{\ln a \cdot x} \cdot \ln a = \ln a \cdot a^x$$

Summary of Differentiation Rules

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^r) = r\,x^{r-1}$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = \ln a \cdot a^x$$

$$(f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(cf)' = cf'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$