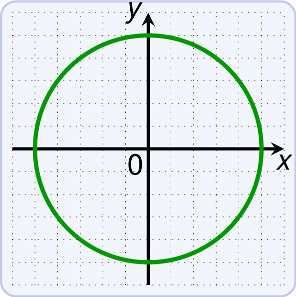
Implicit Differentiation

Consider the equation:

$$x^2 + y^2 = 25$$

This equation describes a circle:



This is not a function and we cannot write it as:

$y = \ldots$ unless we split the circle in upper and lower half

How to compute the slope of points on this curve?

Implicit Differentiation

We can use **implicit differentiation**:

- differentiate both sides of the equation w.r.t. x, and
- then solve for y', that is, for $\frac{dy}{dx}$

We differentiate $x^2 + y^2 = 25$ implicitly. We have

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x + \frac{d}{dx}y^2 = 0 \qquad y \text{ is a function of } x \implies \text{chain rule}$$

$$2x + \frac{d}{dy}(y^2)\frac{d}{dx}y = 0$$

$$2x + 2y\frac{d}{dx}y = 0 \implies \frac{d}{dx}y = -\frac{x}{y} \implies \frac{dy}{dx} = -\frac{x}{y}$$

Implicit Differentiation

We can use **implicit differentiation**:

- differentiate both sides of the equation w.r.t. x, and
- then solve for y', that is, for $\frac{dy}{dx}$

We differentiate $x^2 + y^2 = 25$ implicitly. We have

$$\frac{dy}{dx} = -\frac{x}{y}$$

Find an equation of the tangent at point $(3, 4)$.

At point $(3, 4)$ we have:

$$\frac{dy}{dx} = -\frac{3}{4}$$

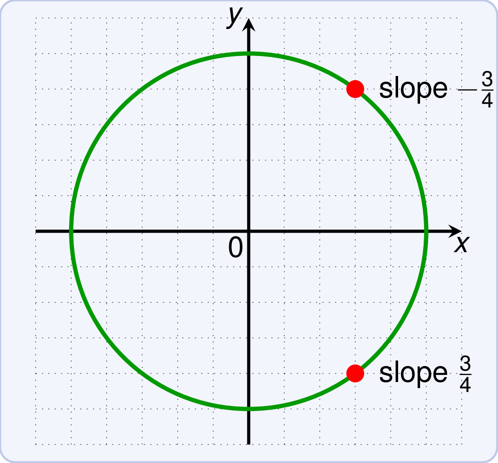
Thus the tangent is

$$y - 4 = -\frac{3}{4}(x - 3)$$

Implicit Differentiation

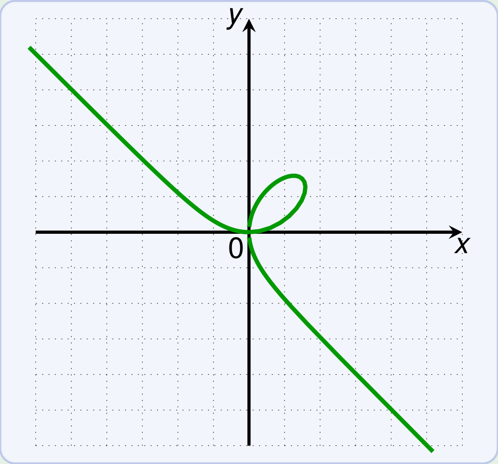
Note that the derivative now depends on x and y!

$$\frac{dy}{dx} = -\frac{x}{y}$$



Implicit Differentiation

Find y' where $x^3 + y^3 = 6xy$.



Implicit Differentiation

Find y' where $x^3 + y^3 = 6xy$.

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}6xy$$

$$3x^2 + 3y^2 \cdot y' = \frac{d}{dx}6xy$$

$$3x^2 + 3y^2 \cdot y' = 6x\frac{d}{dx}y + y\frac{d}{dx}6x$$

$$3x^2 + 3y^2 \cdot y' = 6xy' + 6y$$ we solve for y'

$$3y^2 \cdot y' - 6xy' = +6y - 3x^2$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

Implicit Differentiation

Find y' where $x^3 + y^3 = 6xy$.

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

Find the tangent to the curve at point $(3, 3)$:

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

Thus the tangent is $\quad y - 3 = -1(x - 3)$.

Implicit Differentiation

Find y' where $x^3 + y^3 = 6xy$.

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

At what point in the first quadrant is the tangent horizontal?

In the first quadrant $x > 0$ and $y > 0$, and

$$2y - x^2 = 0 \quad \Longrightarrow \quad y = \frac{x^2}{2} \quad \Longrightarrow \quad x^3 + (\frac{x^2}{2})^3 = 6x\frac{x^2}{2}$$

$$\Longrightarrow \quad \frac{x^6}{8} - 2x^3 = 0 \quad \Longrightarrow \quad x^3(\frac{x^3}{8} - 2) = 0$$

Since $x > 0$, we get $x = \sqrt[3]{16}$. Then the point is

$$(\sqrt[3]{16}, \sqrt[3]{32})$$

Implicit Differentiation

Find y' where

$$\sin(x+y) = y^2 \cos x$$

We have:

$$\begin{aligned}
\frac{d}{dx}\sin(x+y) &= \frac{d}{dx}(y^2\cos x) \\
\cos(x+y)\cdot(1+y') &= \cos x \cdot \frac{d}{dx}(y^2) + y^2\frac{d}{dx}(\cos x) \\
\cos(x+y) + y'\cos(x+y) &= \cos x\cdot(2yy') + y^2(-\sin x) \\
y'\cos(x+y) - 2yy'\cos x &= -y^2\sin x - \cos(x+y) \\
y'(\cos(x+y) - 2y\cos x) &= -(y^2\sin x + \cos(x+y)) \\
y' &= -\frac{y^2\sin x + \cos(x+y)}{\cos(x+y) - 2y\cos x}
\end{aligned}$$

Implicit Differentiation

Find y'' where

$$x^4 + y^4 = 16$$

We have:

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}16 \quad\Longrightarrow\quad 4x^3 + 4y^3y' = 0 \quad\Longrightarrow\quad y' = -\frac{x^3}{y^3}$$

Thus

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\frac{d}{dx}x^3 - x^3\frac{d}{dx}y^3}{(y^3)^2} = -\frac{y^3 3x^2 - x^3 3y^2 y'}{y^6}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} = -\frac{3x^2(x^4 + y^4)}{y^7} = -\frac{48x^2}{y^7}$$