Related (Dependent) Rates

Air is pumped into a spherical balloon:

- the volume increases with 100 cm^3/s

Find: rate of change of the radius when the diameter is $50cm$.

First step: introduce suggestive notation

- let $V(t)$ be the volume after time t
- let $r(t)$ be the radius after time t

Then the given problem translates to

$$V'(t) = 100 \text{ cm}^3/\text{s} \qquad \text{Find } r'(t) \text{ when } r = 25\text{cm}.$$

How are the volume of a sphere and its radius related?

$$V = \frac{4}{3}\pi r^3 \quad \text{thus} \quad V'(t) = \frac{d}{dt}\left(\frac{4}{3}\pi r(t)^3\right) = \frac{4}{3}\pi \cdot 3r(t)^2 r'(t)$$

We solve for $r'(t)$:

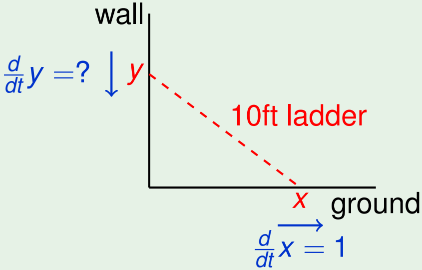
$$r'(t) = \frac{V'(t)}{4\pi \cdot r(t)^2} \qquad r'(t) = \frac{100}{4\pi \cdot 25^2} = \frac{1}{25\pi} \text{ cm/s}$$

Related (Dependent) Rates

A ladder of length 10ft rests against a vertical wall.

- the bottom of the ladder slides away from the wall with 1ft/s

How fast is the top sliding when the bottom is 6ft from the wall?



Thus

$$x^2 + y^2 = 10^2$$
$$\overset{x=6}{\Longrightarrow} \quad 6^2 + y^2 = 10^2$$
$$\Longrightarrow \quad y = \pm\sqrt{10^2 - 6^2}$$
$$\Longrightarrow \quad y = 8$$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}10^2 \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \implies \frac{dy}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4}$$

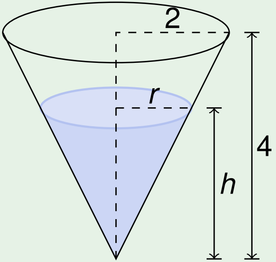
The top slides with $\frac{3}{4}$ft/s when the bottom is 6ft from the wall.

Related (Dependent) Rates

A water tank has the shape of an inverted circular cone:

- base radius $2m$ and the height is $4m$,
- water is pumped into the tank at a rate of $2\text{m}^3/\text{min}$.

At what rate is the water rising when the water is 3m deep?



$$V = \frac{1}{3}\pi r^2 h$$

How is r related to h?

$$\frac{r}{h} = \frac{2}{4} \quad \Longrightarrow \quad r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

We differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) = \frac{1}{12}\pi 3h^2\frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \stackrel{h=3}{=} \frac{4}{\pi 9}\cdot 2$$

Thus the water rises with $8/(\pi 9)$m/min when its is 3m deep.

Related (Dependent) Rates

Problem Solving Strategy

Important when solving textual problems:

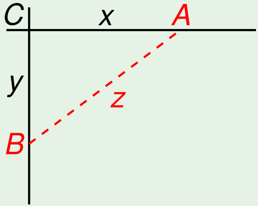
- Read the problem carefully.
- Draw a diagram.
- Introduce notation, function names for the quantities.
- Express given information and goal using the notation.
- Write equations relating the quantities. Eliminate dependent variables (in the previous example we have eliminated the radius as it was dependent on the height).
- Use the chain rule to differentiate both sides w.r.t. t.
- Solve for the unknown rate, and substitute the given information into the resulting formula.

Related (Dependent) Rates

Two cars are headed for the same road intersection:

- car A is traveling west with 50mi/h
- car B is traveling north with 60mi/h

At what rate are the cars approaching
when A is 0.3mi and B is 0.4mi from the intersection?



- $x(t) =$ distance of A to crossing
- $y(t) =$ distance of B to crossing
- $z(t) =$ distance of A to B

$$\frac{d}{dt}x = -50 \qquad \frac{d}{dt}y = -60$$

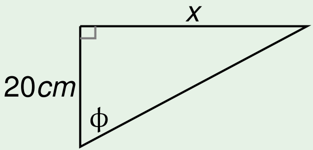
$$z^2 = x^2 + y^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt} \implies \frac{dz}{dt} = \frac{0.3}{0.5}(-50) + \frac{0.4}{0.5}(-60) = -78$$

When $x = 0.3$ & $y = 0.4$, we get $z = 0.5$. The answer is 78mi/h.

Related (Dependent) Rates

We have a right-angled triangle of the form



The length x increases with $4cm/s$.
How fast is the angle ϕ changing when $x = 15cm$?

The quantities x and ϕ are related by:

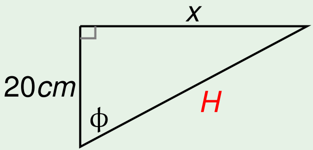
$$\tan\phi = \frac{x}{20}$$

Differentiating both sides yields:

$$\frac{d}{dt}\tan\phi = \frac{d}{dt}\frac{x}{20} \implies \frac{1}{(\cos\phi)^2} \cdot \frac{d\phi}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

Related (Dependent) Rates

We have a right-angled triangle of the form



The length x increases with $4cm/s$.

How fast is the angle ϕ changing when $x = 15cm$?

$$\frac{1}{(\cos\phi)^2} \cdot \frac{d\phi}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

$$\Longrightarrow \quad \frac{d\phi}{dt} = \frac{(\cos\phi)^2}{20} \cdot \frac{dx}{dt} = \frac{(\cos\phi)^2}{20} \cdot 4 = \frac{(\cos\phi)^2}{5}$$

We have $\cos\phi = 20/H = 20/\sqrt{15^2 + 20^2} = 20/25 = 4/5$.

Thus

$$\frac{d\phi}{dt} = \left(\frac{4}{5}\right)^2 \cdot \frac{1}{5} = \frac{4^2}{5^3} = \frac{16}{125}\text{rad/s}$$