L'Hospital's Rule

A limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

is called **indeterminate form of type $\frac{0}{0}$.**

Often cancellation of common factors helps:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)x}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \text{and} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

L'Hospital's Rule

A limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

is called **indeterminate form of type $\frac{\infty}{\infty}$.**

Often helps to divide by highest power of x in the denominator:

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

L'Hospital's Rule

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a, and

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is $-\infty$ or ∞.

(near a = on an open interval containing a except possibly a itself)

Before applying L'Hospital's Rule it is important to verify that:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

L'Hospital's Rule

Find

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

We have

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

and hence we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

L'Hospital's Rule

Find

$$\lim_{x \to 0} \frac{\sin x}{x}$$

We have

$$\lim_{x \to 0} \sin x = 0 \qquad \text{and} \qquad \lim_{x \to 0} x = 0$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

L'Hospital's Rule

Find

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

We have

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to\infty} x^2 = \infty$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Again we have:

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to\infty} 2x = \infty$$

So we can again use l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

L'Hospital's Rule

Find

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$$

We have

$$\lim_{x \to \infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x \to \infty} \sqrt[3]{x} = \infty$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \sqrt[3]{x}} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\frac{1}{3} x^{-\frac{2}{3}}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0$$

L'Hospital's Rule

Find

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$$

If we were to apply l'Hospital's Rule:

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$$

However, this is wrong!

We have $\lim_{x\to\pi^-}(1-\cos x) = 1-(-1) = 2$.

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = 0$$

Before applying l'Hospital's Rule, always check that the limit is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's Rule

L'Hospital's Rule is valid for one-sided limits and limits at infinity:

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} \qquad \lim_{x \to a^+} \frac{f(x)}{g(x)} \qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} \qquad \lim_{x \to -\infty} \frac{f(x)}{g(x)}$$

L'Hospital's Rule

A limit of the form

$$\lim_{x \to a} (f(x)g(x))$$

where

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

is called **indeterminate form of type** $0 \cdot \infty$.

We then rewrite the limit as:

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} \frac{f(x)}{1/g(x)}$$

an indeterminate form of type $\frac{0}{0}$, or as

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

an indeterminate form of type $\frac{\infty}{\infty}$.

L'Hospital's Rule

Evaluate the limit

$$\lim_{x \to 0^+} x \ln x$$

We have

$$\lim_{x \to 0^+} x = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} \ln x = -\infty$$

Thus we can choose for rewriting to:

$$\lim_{x \to 0^+} \frac{x}{1/\ln x} \qquad \text{or} \qquad \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

We choose the 2nd since the derivatives are easier:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

L'Hospital's Rule

A limit of the form

$$\lim_{x \to a} (f(x) - g(x))$$

where

$$\lim_{x \to a} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \infty$$

is called **indeterminate form of type $\infty - \infty$.**

We then rewrite the limit as a **quotient.**

L'Hospital's Rule

Evaluate the limit

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$$

Then $\lim_{x \to (\pi/2)^-} \sec x = \infty$ and $\lim_{x \to (\pi/2)^-} \tan x = \infty$

We use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x}$$

Now $\lim_{x \to (\pi/2)^-} (1 - \sin x) = 0$ and $\lim_{x \to (\pi/2)^-} \cos x = 0$

Hence we can apply l'Hospital's Rule:

$$\begin{aligned} \lim_{x \to (\pi/2)^-} (\sec x - \tan x) &= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} \\ &= \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0 \end{aligned}$$

L'Hospital's Rule

A limit of the form

$$\lim_{x\to a}[f(x)]^{g(x)}$$

is an indeterminate form

- **of type** 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
- **of type** ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$
- **of type** 1^∞ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

Each of these cases can be treated by writing the limit as:

$$\begin{aligned}\lim_{x\to a}[f(x)]^{g(x)} &= \lim_{x\to a} e^{\ln\left([f(x)]^{g(x)}\right)} \\ &= \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a}(g(x)\ln f(x))}\end{aligned}$$

Other types are **not** indeterminate forms: 0^∞, 1^0 and ∞^1.

L'Hospital's Rule

Evaluate the limit

$$\lim_{x \to 0^+} x^x$$

Then $\lim_{x \to 0^+} x = 0$.

We write the limit as:

$$\begin{aligned} \lim_{x \to 0^+} x^x &= \lim_{x \to 0^+} e^{\ln x^x} \\ &= e^{\lim_{x \to 0^+} (x \ln x)} \\ &= e^0 \\ &= 1 \end{aligned}$$

L'Hospital's Rule

Evaluate the limit $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

Then $\lim_{x \to 0^+} (1 + \sin 4x) = 1$ and $\lim_{x \to 0^+} \cot x = \infty$

We write the limit as:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{cotx} = \lim_{x \to 0^+} e^{\ln(1+\sin 4x)^{cotx}}$$
$$= e^{\lim_{x \to 0^+} (\cot x \cdot \ln(1+\sin 4x))}$$

$$\lim_{x \to 0^+} (\cot x \cdot \ln(1 + \sin 4x)) = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

Now $\lim_{x \to 0^+} \ln(1 + \sin 4x) = 0$ and $\lim_{x \to 0^+} \tan x = 0$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\frac{4 \cos 4x}{1+\sin 4x}}{(\sec x)^2} = \frac{\left(\frac{4}{1}\right)}{1} = 4$$

Thus $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = e^4$