Curve Sketching

For sketching a curve of $f(x)$:

- determine the **domain**
- find the y**-intercept** $f(0)$ and the x**-intercepts** $f(x) = 0$
- find **vertical asymptotes** $x = a$, that is:
  $$\lim_{x \to a^-} = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} = \pm\infty$$
- find **horizontal asymptotes** $y = L$, that is:
  $$\lim_{x \to \infty} = L \qquad \text{or} \qquad \lim_{x \to -\infty} = L$$
- find intervals of **increase** $f'(x) > 0$ and **decrease** $f'(x) < 0$
- find **local maxima and minima**
- determine **concavity** on intervals and **points of inflection**
  - $f''(x) > 0$ concave upward
  - $f''(x) < 0$ concave downward
  - inflections points where $f''(x)$ changes the sign

Curve Sketching

For local minima and maxima:

- find critical numbers c
- then the first First Derivative Test:
  - f' changes from $+$ to $-$ at $c \implies$ maximum
  - f' changes from $-$ to $+$ at $c \implies$ minimum
- Second Derivative Test:
  - $f''(c) < 0 \implies$ maximum
  - $f''(c) > 0 \implies$ minimum
  - $f''(c) = 0 \implies$ use First Derivative Test

Then sketch the curve:

- draw asymptotes as thin dashed lines
- mark intercepts, local extrema and inflection points
- draw the curve taking into account:
  - increase / decrease, concavity and asymptotes

Curve Sketching

Sketch the curve of $f(x) = \frac{2x^2}{x^2-1}$.

The domain is $\{x \mid x \neq \pm 1\}$, that is, $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

We have $\quad f(0) = 0 \quad$ and $\quad f(x) = 0 \iff x = 0$

The vertical asymptotes are $\quad x = -1 \quad$ and $\quad x = 1$

$$\lim_{x \to -1^-} = \infty \qquad \lim_{x \to -1^+} = -\infty \qquad \lim_{x \to 1^-} = -\infty \qquad \lim_{x \to 1^+} = \infty$$

The horizontal asymptotes are $\quad y = 2$

$$\lim_{x \to \infty} f(x) = 2 \qquad \lim_{x \to -\infty} f(x) = 2$$

Curve Sketching

Sketch the curve of $f(x) = \frac{2x^2}{x^2-1}$.

The derivative is:

$$f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Thus

- increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (-1, 0)$
- decreasing on ($f'(x) < 0$) on $(0, 1) \cup (1, \infty)$

The critical numbers are $x = 0$ (since $f'(0) = 0$)

- $f'(x)$ changes from $+$ to $-$ at 0 $\implies$ local maximum $(0, 0)$

Curve Sketching

Sketch the curve of $f(x) = \frac{2x^2}{x^2-1}$.

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$

The second derivative is:

$$\begin{aligned} f''(x) &= \frac{-4(x^2-1)^2 - (-4x)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} \\ &= \frac{-4(x^2-1)+16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3} \end{aligned}$$

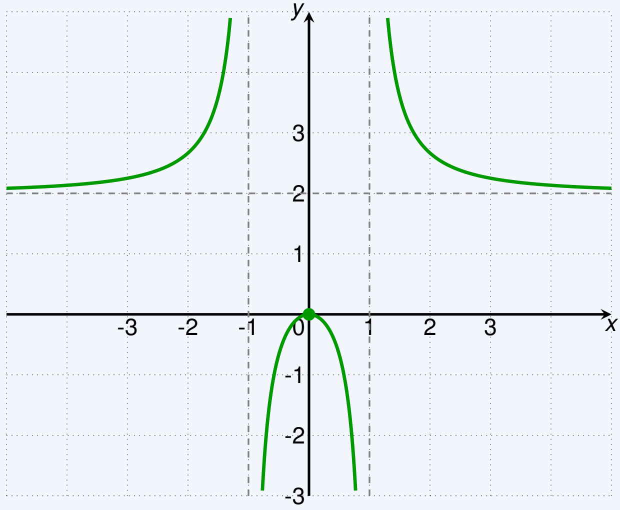
$12x^2 + 4 > 0 \quad \text{for all } x$

$f''(x) > 0 \iff (x^2-1)^3 > 0 \iff x^2 - 1 > 0 \iff |x| > 1$

- concave upward on $(-\infty, -1) \cup (1, \infty)$
- concave downward on $(-1, 1)$
- inflection points: none (-1 and 1 not in the domain)

Curve Sketching

Sketch the curve of $f(x) = \frac{2x^2}{x^2-1}$.



Slant Asymptotes

Asymptotes that are neither horizontal nor vertical:

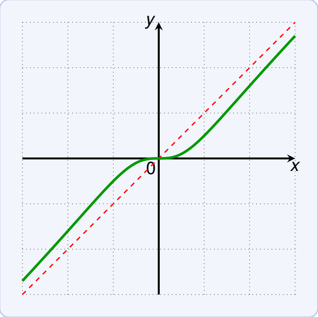
If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

or

$$\lim_{x \to -\infty} [f(x) - (mx + b)] = 0$$

the the line $y = mx + b$ is called **slant asymptote.**



Note that the distance between curve and line approaches 0.

Slant Asymptotes

Sketch the graph of $f(x) = \frac{x^3}{2x^2+1}$.

The domain is $(-\infty, \infty)$

The $\quad f(0) = 0 \quad$ and $\quad f(x) = 0 \iff x = 0$

Vertical asymptotes: none. Horizontal asymptotes: none

Slant asymptotes: $\quad y = \frac{1}{2}x \quad$ since

$$\lim_{x\to\infty}\left(\frac{x^3}{2x^2+1} - \frac{x}{2}\right) = \lim_{x\to\infty}\left(\frac{2x^3 - x(2x^2+1)}{2(2x^2+1)}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-x}{2(2x^2+1)}\right) = 0$$

Slant Asymptotes

Sketch the graph of $f(x) = \frac{x^3}{2x^2+1}$.

$$f'(x) = \frac{3x^2(2x^2+1) - x^3(4x)}{(2x^2+1)^2} = \frac{2x^4+3x^2}{(2x^2+1)^2} = \frac{x^2(2x^2+3)}{(2x^2+1)^2}$$

Thus $f'(x) > 0$ for all $x \neq 0$. Hence increasing on $(-\infty, \infty)$.

Local minima, maxima: none (since f' does not change sign)

We have

$$f''(x) = -\frac{2x(2x^2-3)}{(2x^2+1)^3}$$

Thus $f''(x) = 0 \iff x = 0$ or $x = \pm\sqrt{3/2}$

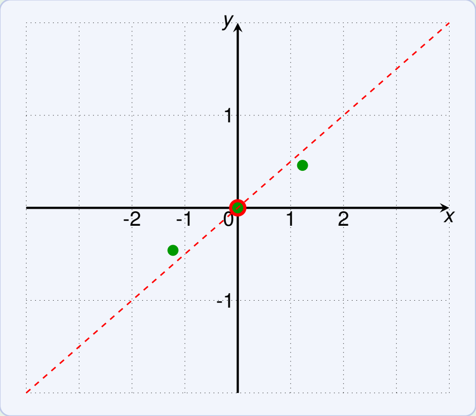
Interval	$f''(x)$	
$x < -\sqrt{3/2}$	+	concave up on $(-\infty, -\sqrt{3/2})$
$-\sqrt{3/2} < x < 0$	-	concave down on $(-\sqrt{3/2}, 0)$
$0 < x < \sqrt{3/2}$	+	concave up on $(0, \sqrt{3/2})$
$\sqrt{3/2} < x$	-	concave up down $(\sqrt{3/2}, \infty)$

Inflection points: $(-\sqrt{\frac{3}{2}}, -\frac{3}{8}\sqrt{\frac{3}{2}})$, $(0,0)$ and $(\sqrt{\frac{3}{2}}, \frac{3}{8}\sqrt{\frac{3}{2}})$

Slant Asymptotes

Sketch the graph of $f(x) = \frac{x^3}{2x^2+1}$.

- x- and y-intercept: $(0, 0)$
- inflection points: $(-\sqrt{\frac{3}{2}}, -\frac{3}{8}\sqrt{\frac{3}{2}})$, $(0, 0)$ and $(\sqrt{\frac{3}{2}}, \frac{3}{8}\sqrt{\frac{3}{2}})$
- slant asymptote: $y = \frac{1}{2}x$



Slant Asymptotes

Sketch the graph of $f(x) = \frac{x^3}{2x^2+1}$.

- increasing on $(-\infty, \infty)$ and $f'(0) = 0$
- concave up on $(-\infty, -\sqrt{3/2})$ and $(0, \sqrt{3/2})$
- concave down on $(-\sqrt{3/2}, 0)$ and $(\sqrt{3/2}, \infty)$