Chapter 4: Motion in Two Dimensions

PHY0101/PHY(PEN)101

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Outline

Position, Velocity, Acceleration

- Just as in 1d, in 2d, an object's motion is completely known if it's position, velocity, and acceleration are known.
- **Position Vector $\equiv$ r**
  - In terms of unit vectors discussed last time, for an object at position **(x,y)** in **x-y** plane:

$$\mathbf{r} \equiv x\,\mathbf{i} + y\,\mathbf{j}$$

For an object moving: **r** depends on time **t:**

$$\mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

- Suppose that an object moves from point **A**(**r_i**) to point **B**(**r_f**) in the **x-y** plane:
- **The Displacement Vector is: ≡ $\Delta r = r_f - r_i$**

  If this happens in in a time

  $$\Delta t = t_f - t_i$$
- **The Average Velocity is:**

  $$v_{avg} \equiv (\Delta r/\Delta t)$$

  Obviously, this in the same direction as the displacement. It is independent of the path between **A** and **B**

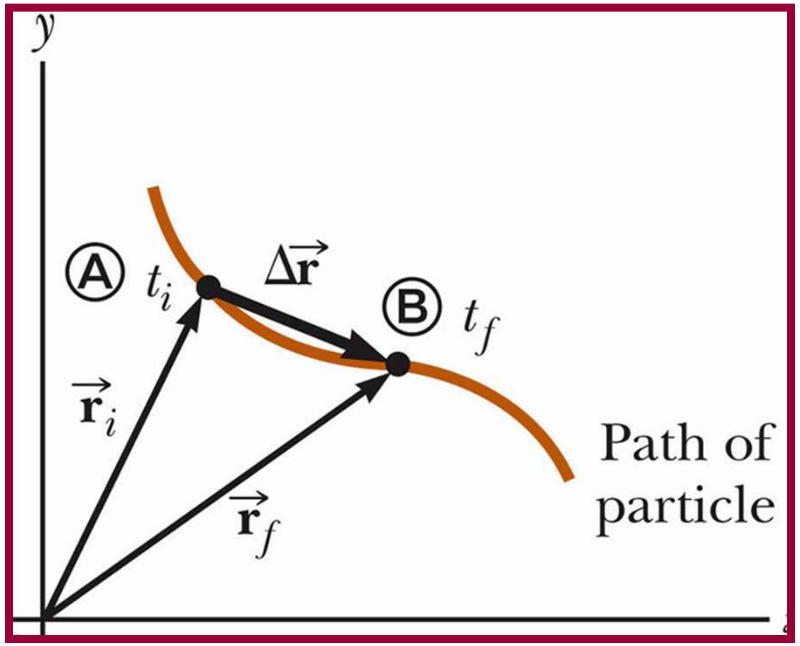


Figure 4.1 A particle moving in the xy plane is located with the position vector $\vec{r}$ drawn from the origin to the particle. The displacement of the particle as it moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$ is equal to the vector $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$.

- As Δt gets smaller & smaller, clearly, **A** and **B** get closer & closer together. Just as in 1d, we define ***The Instantaneous Velocity***

≡ Velocity at Any Instant of Time.

≡ average velocity over an infinitesimally short time

- Mathematically, the instantaneous velocity is:

$$v \equiv \lim_{\Delta t \to 0}[(\Delta r)/(\Delta t)] \equiv (dr/dt)$$

$\lim_{\Delta t \to 0} \equiv$ ratio $(\Delta r)/(\Delta t)$ for smaller & smaller Δt. ***Mathematicians call this a derivative.***

⇒ The instantaneous velocity

$v \equiv$ ***time derivative of displacement*** r

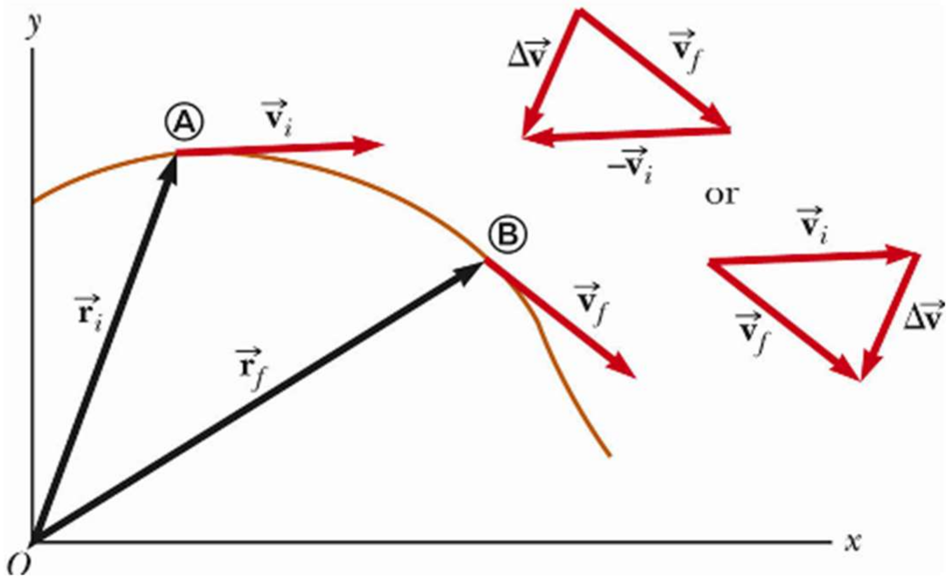
- ***Instantaneous Velocity*** $\mathbf{v} \equiv (d\mathbf{r}/dt)$.
- The magnitude $|v|$ of vector $v \equiv$ **speed.** As motion progresses, the speed & direction of v can both change. For an object moving from **A** (v_i) to **B** (v_f) in the x-y plane:

  Velocity Change $\equiv \Delta v = v_f - v_i$

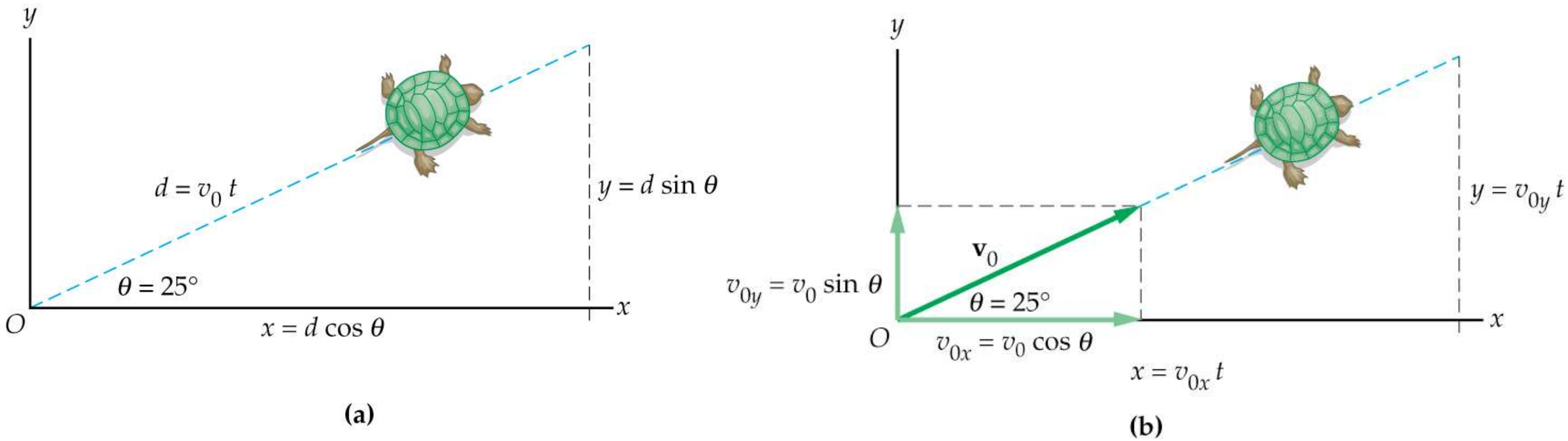
  This happens in time $\Delta t = t_f - t_i$
- **Average Acceleration**

  $$a_{avg} \equiv (\Delta v/\Delta t)$$

  As both the speed & direction of **v** change, over an arbitrary path



Example: Motion of a Turtle

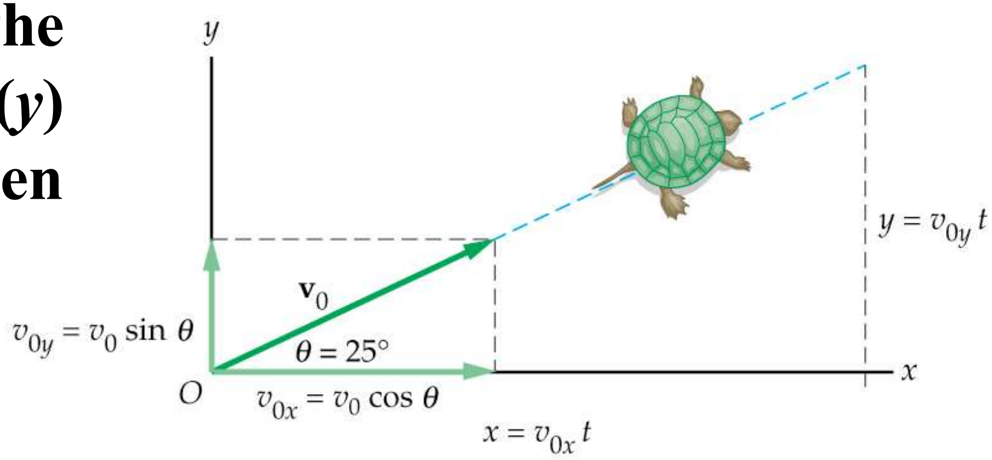


A turtle starts at the origin and moves with the speed of v_0=10 cm/s in the direction of 25° to the horizontal. cos25°=0.906, sin25°=0.423

(a) Find the coordinates of a turtle 10 seconds later.

(b) How far did the turtle walk in 10 seconds?

Notice, you can solve the equations independently for the horizontal (*x*) and vertical (*y*) components of motion and then combine them!



(b)

$$\vec{v}_0 = \vec{v}_x + \vec{v}_y$$

- X components:

$$v_{0x} = v_0 \cos 25^\circ = 9.06 \text{ cm/s}$$

$$\Delta x = v_{0x}t = 90.6 \text{ cm}$$

- Y components:

$$v_{0y} = v_0 \sin 25^\circ = 4.23 \text{ cm/s}$$

$$\Delta y = v_{0y}t = 42.3 \text{ cm}$$

$$\vec{r} = 90.6\hat{\imath} + 42.3\hat{\jmath}\,\text{cm}$$

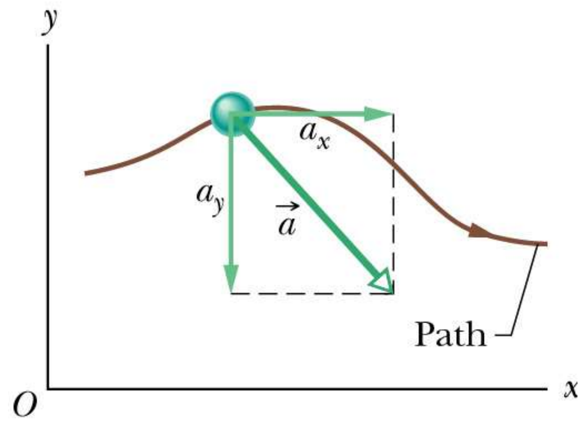
- Distance from the origin:

$$d = \sqrt{\Delta x^2 + \Delta y^2} = 100.0 \text{ cm}$$

Average and Instantaneous Acceleration

- Average acceleration

$$\vec{a}_{avg} \equiv \frac{\Delta\vec{v}}{\Delta t}$$



$$\vec{a}_{avg} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} = a_{avg,x}\hat{i} + a_{avg,y}\hat{j}$$

- Instantaneous acceleration

$$\vec{a} \equiv \lim_{t\to 0}\vec{a}_{avg} = \lim_{t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j}$$

- The magnitude of the velocity (the speed) can change
- The direction of the velocity can change, even though the magnitude is constant
- Both the magnitude and the direction can change

Summary in 2D motion

- Position $\boxed{\vec{r}(t) = x\hat{i} + y\hat{j}}$
- Average velocity $\boxed{\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} = v_{avg,x}\hat{i} + v_{avg,y}\hat{j}}$
- Instantaneous velocity $\boxed{v_x \equiv \frac{dx}{dt}}$ $\boxed{v_y \equiv \frac{dy}{dt}}$

$$\boxed{\vec{v}(t) = \lim_{t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}}$$

- Acceleration $\boxed{a_x \equiv \frac{dv_x}{dt} = \frac{d^2x}{dt^2}}$ $\boxed{a_y \equiv \frac{dv_y}{dt} = \frac{d^2y}{dt^2}}$

$$\boxed{\vec{a}(t) = \lim_{t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j}}$$

2D Motion with Constant Acceleration

- It can be shown that:

Motion in the x-y plane can be treated as 2 independent motions in the x and y directions.

⇒ So, motion in the x direction doesn’t affect the y motion and motion in the y direction doesn’t affect the x motion.

Motion in two dimensions

- Motions in each dimension are independent components
- Constant acceleration equations

$$\vec{v} = \vec{v}_0 + \vec{a}t \qquad \vec{r} - \vec{r} = \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

- Constant acceleration equations hold in each dimension

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t$$

$$x - x_0 = v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad y - y_0 = v_{0y}t + \tfrac{1}{2}a_y t^2$$

$$v_x{}^2 = v_{0x}{}^2 + 2a_x(x - x_0) \qquad v_y{}^2 = v_{0y}{}^2 + 2a_y(y - y_0)$$

  - $t = 0$ beginning of the process;
  - $\vec{a} = a_x\hat{i} + a_y\hat{j}$ where a_x and a_y are constant;
  - Initial velocity $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ initial displacement $\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$

Example 4.1: A particle starts from the origin at *t=0* with an initial velocity having an *x* component of 20 m/s and a *y* component of -15 m/s. The particle moves in the *xy* plane with an *x* component of acceleration only, given by a_x=4.0 m/s^2. (a) Determine the components of the velocity vector at any time and the total velocity vector at any time.

$$v_{xf} = v_{xi} + a_x t = (20 + 4.0t)\ \text{m/s}$$

$$v_{yf} = v_{yi} + a_y t = -15\ \text{m/s} + 0 = -15\ \text{m/s}$$

$$\mathbf{v}_f = v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}} = [(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s}$$

$$\mathbf{a} = 4.0\hat{\mathbf{i}}\ \text{m/s}^2 \qquad \mathbf{v}_i = [20\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s}$$

(b) Calculate the velocity and speed of the particle at t=5.0 s.

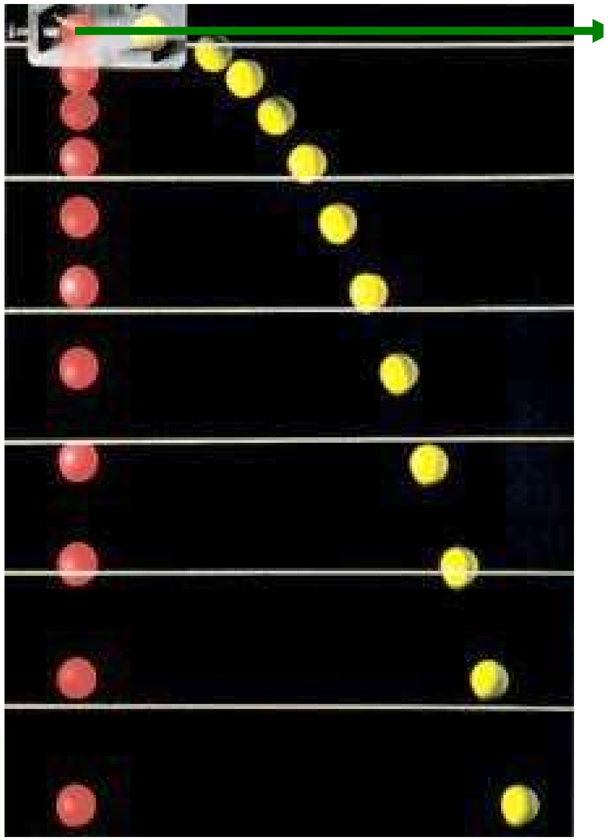
$$\mathbf{v}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s} = (40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\ \text{m/s}$$

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = -21°$$

4.3 Projectile Motion

- 2-D problem and define a coordinate system: x- horizontal, y- vertical (up +)
- Try to pick $x_0 = 0, y_0 = 0$ at $t = 0$
- Horizontal motion + Vertical motion
- Horizontal: $a_x = 0$, constant velocity motion
- Vertical: $a_y = -g = -9.8$ m/s^2, $v_{0y} = 0$
- Equations:

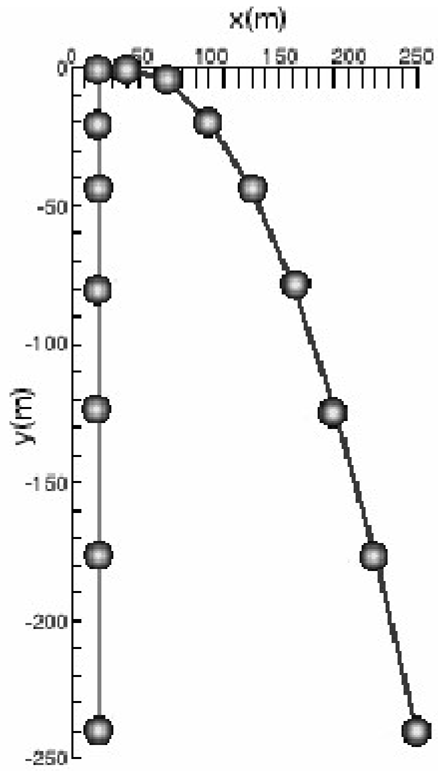
Horizontal	Vertical
$v_x = v_{0x} + a_x t$	$v_y = v_{0y} + a_y t$
$x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2$	$y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2$
$v_x^{\,2} = v_{0x}^{\,2} + 2a_x(x - x_0)$	$v_y^{\,2} = v_{0y}^{\,2} + 2a_y(y - y_0)$



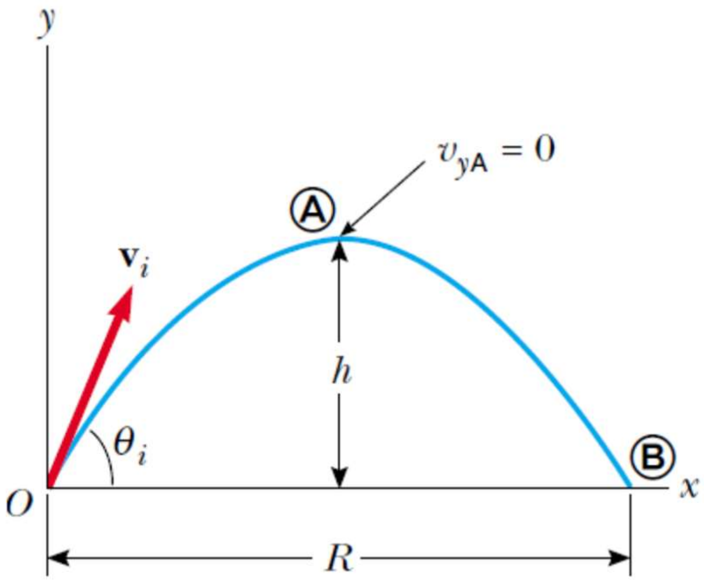
- X and Y motions happen independently, so we can treat them separately

Horizontal	Vertical
$v_x = v_{0x}$	$v_y = v_{0y} - gt$
$\vartheta_{xi} = \vartheta_i cos\theta_i$	$\vartheta_{yi} = \vartheta_i sin\theta_i$
$x = x_0 + v_{0x}t$	$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$

- Try to pick $x_0 = 0,\ y_0 = 0$ at $t = 0$
- Horizontal motion + Vertical motion
- Horizontal: $a_x = 0$, constant velocity motion
- Vertical: $a_y = -g = -9.8\ \mathrm{m/s^2}$
- x and y are connected by time t
- y(x) is a parabola



Maximum Height of a Projectile



$$v_{yf} = v_{yi} + a_y t$$

$$0 = v_i \sin\theta_i - g t_A$$

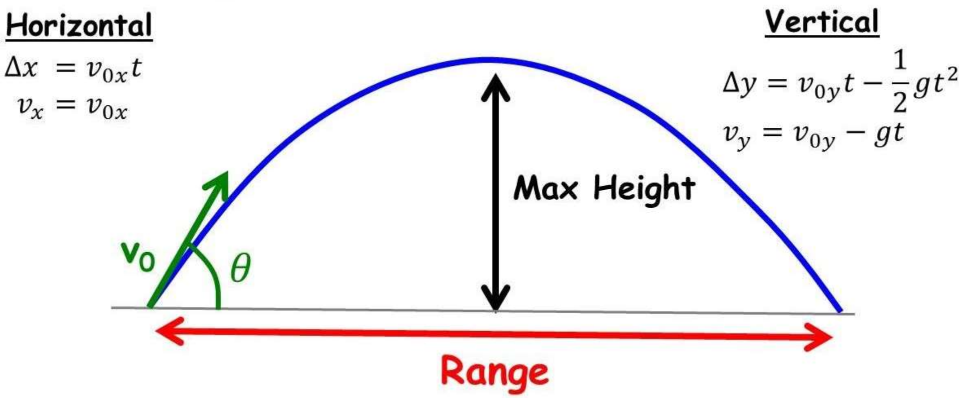
$$t_A = \frac{v_i \sin\theta_i}{g}$$

$$h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$\boldsymbol{h = \frac{\vartheta_i^2 \sin^2\theta_i}{2g}}$$

Maximum Height $\boxed{h = \frac{\vartheta_{yi}^2}{2g}}$

Horizontal Range of a Projectile



$$t = 2t_A = \frac{2\vartheta_i sin\theta_i}{g}$$

$$R = \vartheta_{xi} t = \vartheta_i cos\theta_i \frac{2\vartheta_i sin\theta_i}{g} = \frac{2\vartheta_i^2 sin\theta_i cos\theta_i}{g}$$

$$\boxed{R = \frac{2\vartheta_{xi}\vartheta_{yi}}{g}}$$

Example 4.3: The Long Jump

A long-jumper leaves the ground at an angle $\theta_i = 20^\circ$ above the horizontal at a speed of $v_i = 11.0$ m/s.(cos20°=0.94, sin20°=0.34)

a) How far does he jump in the horizontal direction?
(Assume his motion is equivalent to that of a particle.)

b) What is the maximum height reached?

<u>**Kinematic Equations**</u>

$$v_{xi} = v_i\cos\theta_i, \quad v_{yi} = v_i\sin\theta_i, v_{xf} = v_{xi}$$
$$x_f = v_{xi}\, t, \; v_{yf} = v_{yi} - gt$$
$$y_f = v_{yi}\, t - (\tfrac{1}{2})gt^2$$
$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

The Long Jump: Solutions

A long-jumper leaves the ground at an angle $\theta_i = 20^\circ$ above the horizontal at a speed of $v_i = 8.0$ m/s. ($\cos 20° = 0.94$, $\sin 25° = 0.34$)

a) How far does he jump in the horizontal direction?

(Assume his motion is equivalent to that of a particle.)

b) What is the maximum height reached?

$$v_{xi} = v_i \cos(\theta_i) = 7.5 \text{ m/s}$$

$$v_{yi} = v_i \sin(\theta_i) = 4.0 \text{ m/s}$$

a) How far does he jump in the horizontal direction? Range =

$$R = (2 v_{xi} v_{yi}/g) = 2(7.5)(4)/(9.8)$$

$$R = 7.94 \text{ m}$$

Kinematic Equations

$$v_{xi} = v_i \cos\theta_i, \quad v_{yi} = v_i \sin\theta_i, v_{xf} = v_{xi}$$

$$x_f = v_{xi}\, t, \; v_{yf} = v_{yi} - gt$$

$$y_f = v_{yi}\, t - (½) g t^2$$

$$(v_{yf})^2 = (v_{yi})^2 - 2 g y_f$$

The Long Jump: Solutions

A long-jumper leaves the ground at an angle $\theta_i = 20^\circ$ above the horizontal at a speed of $v_i = 11.0$ m/s. (cos20°=0.94, sin20°=0.34)

a) How far does he jump in the horizontal direction? (Assume his motion is equivalent to that of a particle.)

b) What is the maximum height reached?

$$v_{xi} = v_i \cos(\theta_i) = 7.5 \text{ m/s}$$

$$v_{yi} = v_i \sin(\theta_i) = 4.0 \text{ m/s}$$

$$R = 7.94 \text{ m}$$

b) What is the maximum height?

$$h = [(v_{yi})^2/(2g)]$$

$$h = 0.72 \text{ m}$$

Kinematic Equations

$$v_{xi} = v_i\cos\theta_i, \quad v_{yi} = v_i\sin\theta_i, v_{xf} = v_{xi}$$

$$x_f = v_{xi}\, t, \; v_{yf} = v_{yi} - gt$$

$$y_f = v_{yi}\, t - (½)gt^2$$

$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

Example 4.4: Non-Symmetric Projectile Motion

A stone is thrown. $x_i = y_i = 0$

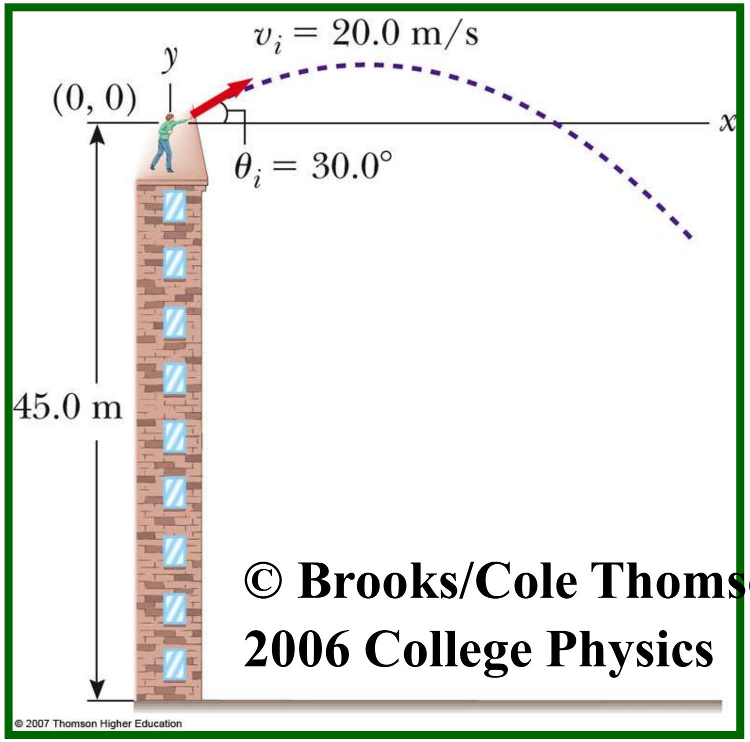
$y_f = -45.0$ m, $v_i = 20$ m/s, $\theta_i = 30^o$

a) Time to hit the ground?

b) Speed just before it hits?

c) Distance from the base of the building where it lands?

(cos30°=0.87, sin30°=0.5)



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Kinematic Equations

$$v_{xi} = v_i\cos\theta_i, \quad v_{yi} = v_i\sin\theta_i$$

$$v_{xf} = v_{xi}, \quad x_f = v_{xi}\,t$$

$$v_{yf} = v_{yi} - gt$$

$$y_f = v_{yi}\,t - (½)gt^2$$

$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

Example 4.4: Solution

A stone is thrown! $x_i = y_i = 0$

$y_f = -45.0$ m, $v_i = 20$ m/s, $\theta_i = 30^o$

a) Time to hit the ground?

b) Speed just before it hits?

c) Distance from the base of the building where it lands?

First, calculate

$$v_{xi} = v_i \cos(\theta_i) = 17.3 \text{ m/s}$$
$$v_{yi} = v_i \sin(\theta_i) = 10.0 \text{ m/s}$$

a) Time to hit the ground?

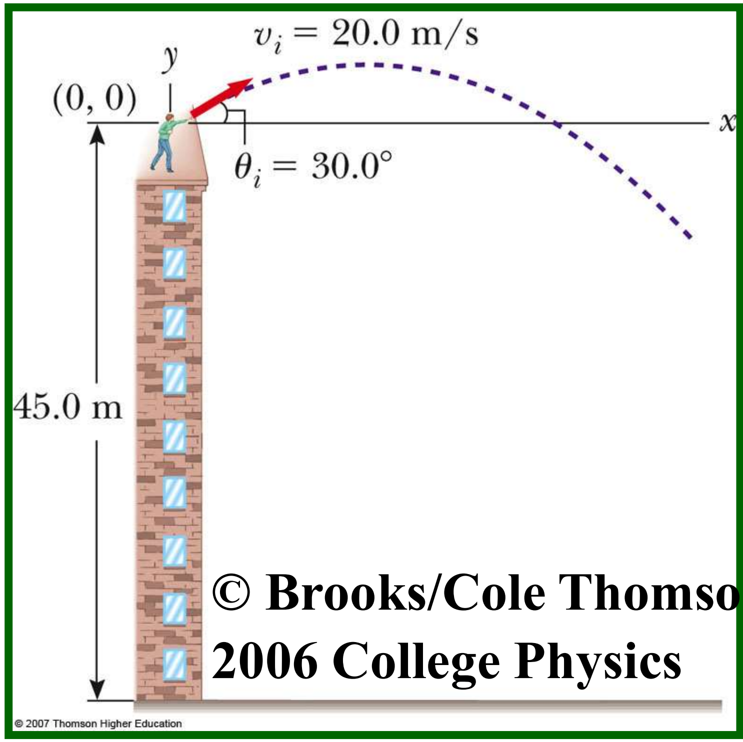
(**Time when $y_f = -45.0$ m**)

$$y_f = -45\text{m} = v_{yi}\, t - (½)gt^2$$

A general quadratic must be solved using the quadratic equation.

This gives:

$$t = 4.22 \text{ s}$$



Kinematic Equations

$$v_{xi} = v_i\cos\theta_i, \quad v_{yi} = v_i\sin\theta_i$$
$$v_{xf} = v_{xi}, \quad x_f = v_{xi}\, t$$
$$v_{yf} = v_{yi} - gt$$
$$y_f = v_{yi}\, t - (½)gt^2$$
$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

Example 4.4: Solution

A stone is thrown! $x_i = y_i = 0$

$y_f = -45.0$ m, $v_i = 20$ m/s, $\theta_i = 30^o$

a) Time to hit the ground?

b) Speed just before it hits?

c) Distance from the base of the building where it lands?

First, calculate

$$v_{xi} = v_i \cos(\theta_i) = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin(\theta_i) = 10.0 \text{ m/s}$$

$$t_{hit} = 4.22 \text{ s}$$

b) Velocity just before it hits?

$v_{xf} = v_{xi}$, $v_{yf} = v_{yi} - gt$ so $v_{xf} = 17.3$ m/s

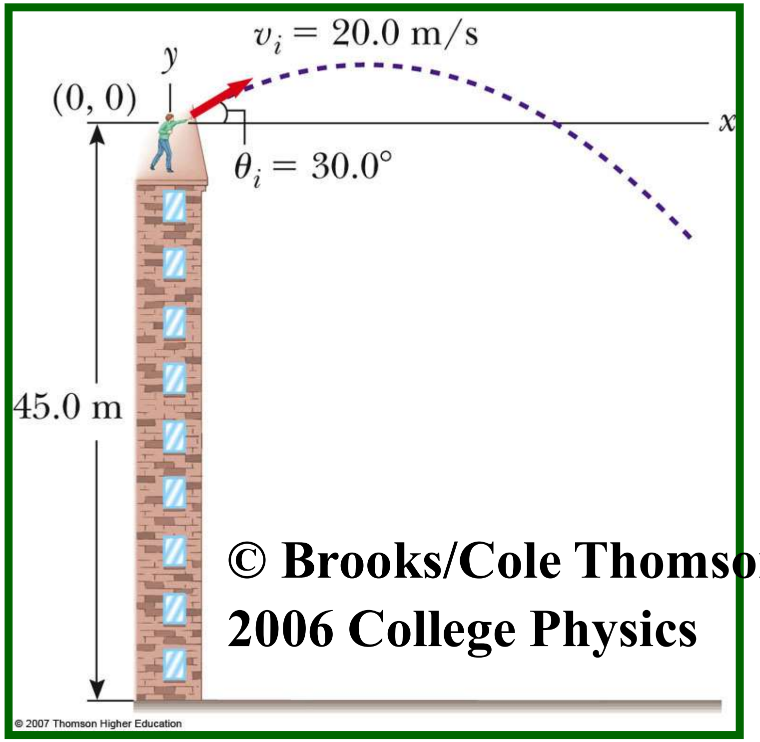
$v_{yf} = 10 - (9.8)(4.22) = -31.3$ m/s

Speed $(v_f)^2 = (v_{xf})^2 + (v_{yf})^2$

$$v_f = 35.8 \text{ m/s}$$

Angle: $\tan(\theta_f) = (v_{yf}/v_{xf}) = -(31.3/17.3) = -1.8$

$$\theta_f = -60.9^o$$



Kinematic Equations

$v_{xi} = v_i\cos\theta_i$, $v_{yi} = v_i\sin\theta_i$

$v_{xf} = v_{xi}$, $x_f = v_{xi}\, t$

$v_{yf} = v_{yi} - gt$

$y_f = v_{yi}\, t - (½)gt^2$

$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$

Example 4.4: Solution

A stone is thrown! $x_i = y_i = 0$

$y_f = -45.0$ m, $v_i = 20$ m/s, $\theta_i = 30^o$

a) Time to hit the ground?

b) Speed just before it hits?

c) Distance from the base of the building where it lands?

First, calculate

$$v_{xi} = v_i \cos(\theta_i) = 17.3 \text{ m/s}$$

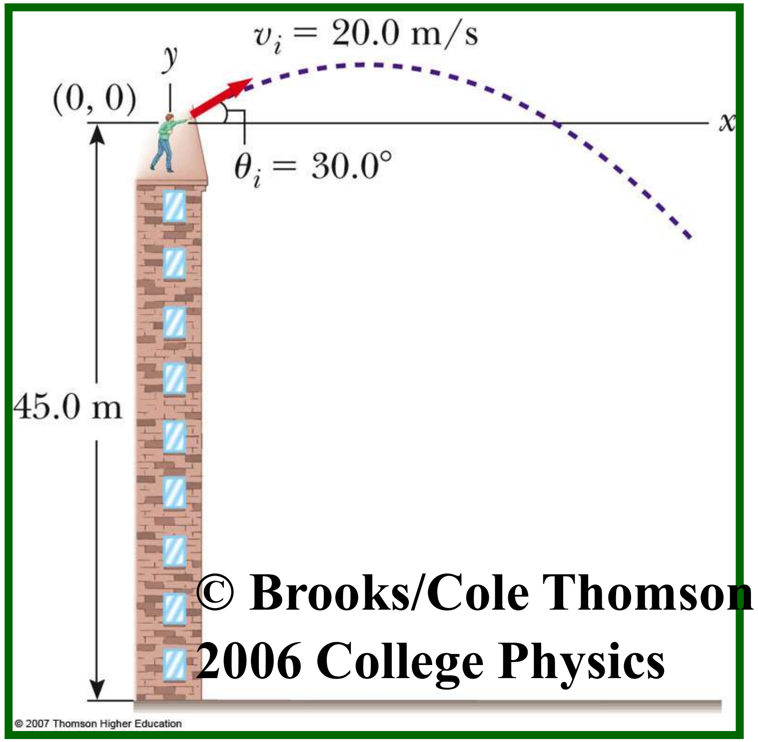
$$v_{yi} = v_i \sin(\theta_i) = 10.0 \text{ m/s}$$

$$t_{hit} = 4.22 \text{ s}$$

$$v_f = 35.8 \text{ m/s}, \theta_f = -60.9^o$$

c) Distance from the base of the building where it lands?

$$x_f = v_{xi} t_{hit} = (17.3)(4.22) = 73.0 \text{ m}$$



Kinematic Equations

$$v_{xi} = v_i\cos\theta_i, \quad v_{yi} = v_i\sin\theta_i$$

$$v_{xf} = v_{xi}, \quad x_f = v_{xi}\, t$$

$$v_{yf} = v_{yi} - gt$$

$$y_f = v_{yi}\, t - (½)gt^2$$

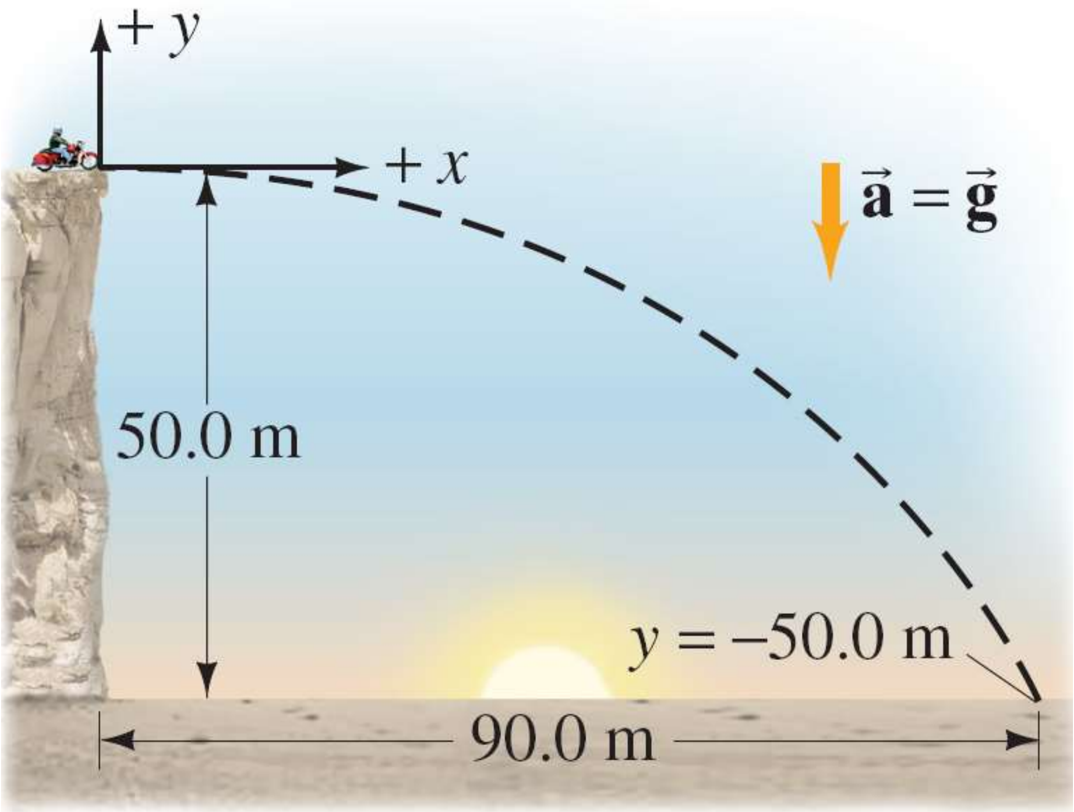
$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

Example: Driving Off a Cliff!!

A movie stunt driver on a motorcycle speeds horizontally off a **50.0-m**-high cliff. How fast must the motorcycle leave the cliff top to land on level ground below, **90.0 m** from the base of the cliff where the cameras are?

Kinematic Equations: $v_{xi} = v_i \cos\theta_i$, $v_{yi} = v_i \sin\theta_i$, $v_{xf} = v_{xi}$ $x_f = v_{xi} t$

$v_{yf} = v_{yi} - gt$, $y_f = v_{yi} t - (½)gt^2$, $(v_{yf})^2 = (v_{yi})^2 - 2gy_f$

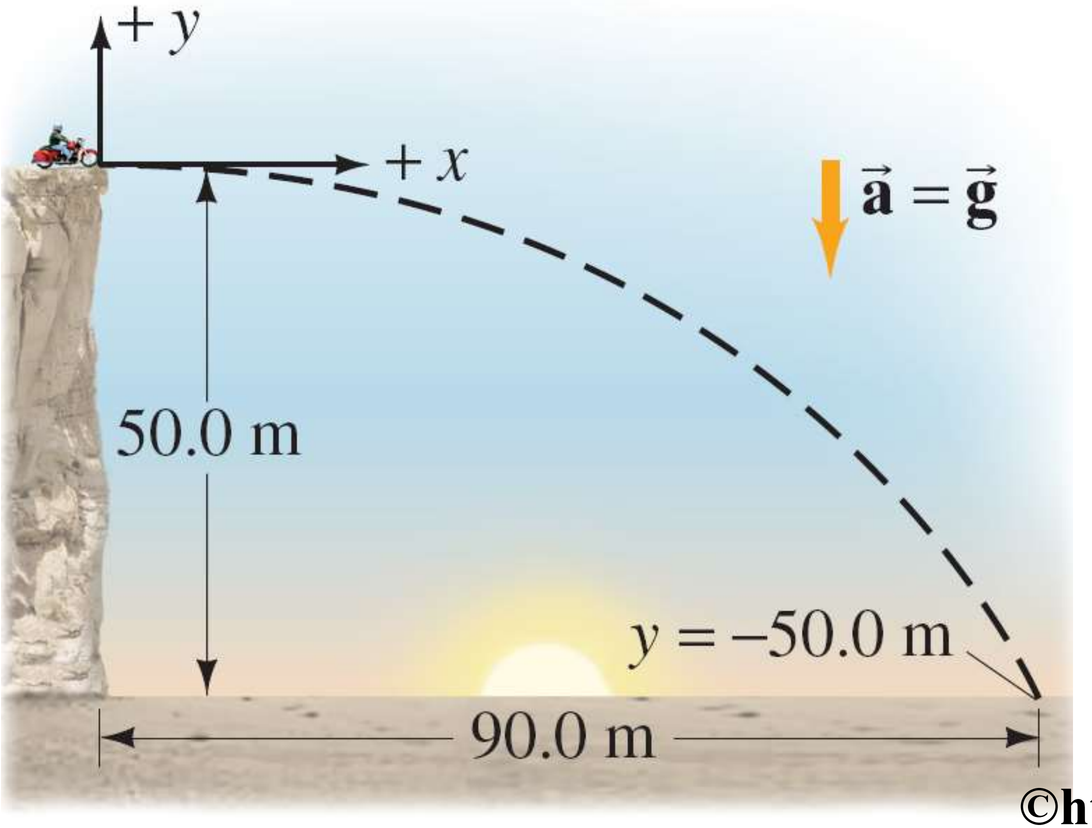


Solutions: Driving Off a Cliff!!

A movie stunt driver on a motorcycle speeds horizontally off a **50.0-m**-high cliff. How fast must the motorcycle leave the cliff top to land on level ground below, **90.0 m** from the base of the cliff where the cameras are?

Kinematic Equations: $v_{xi} = v_i\cos\theta_i$, $v_{yi} = v_i\sin\theta_i$, $v_{xf} = v_{xi}$, $x_f = v_{xi}t$

$v_{yf} = v_{yi} - gt$, $y_f = v_{yi}t - (\frac{1}{2})gt^2$, $(v_{yf})^2 = (v_{yi})^2 - 2gy_f$



$v_x = v_{xi} = ?$, $v_{yf} = -gt$

$x_f = v_{xi}t$, $y_f = -(\frac{1}{2})gt^2$

Time to the bottom = time when $y = -50$ m

$-(\frac{1}{2})gt^2 = -50$ m

$t = 3.19$ s

At that time $x_f = 90.0$ m

So $v_{xi} = (x_f/t) = (90/3.19)$

$v_{xi} = 28.2$ m/s