CHAPTER 1. CLASSIFICATION of DIFFERENTIAL EQUATIONS

Definition. An equation involving derivatives of one or more independent variables is called a differential equation**.**

Example.

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^3 = 0 \tag{1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = e^x \tag{2}$$

Definition. The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.
Equation (1) is a second order ordinary differential equation. Equation (2) is a fourth order ordinary differential equation.

Definition. A linear ordinary differential equation of order n, in the independent variable y and the dependent variable x, is an equation that can be expressed in the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x), \tag{3}$$

where a_0 is not identically zero.

Definition. An equation which is not linear is called a nonlinear differential equation.

Example. The following differential equations are linear:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0 \tag{4}$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^2y}{dx^2} - e^x\frac{dy}{dx} + 4y = \cos x \tag{5}$$

The following differential equations are nonlinear:

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 - 2y = 0$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^2y}{dx^2} - e^x\frac{dy}{dx} + 4y = \cos y$$

Definition. In equation (3), if one of the coefficients $a_0(x),\ a_1(x), ..., a_n(x)$ depend on x, then equation (3) is said that linear with variable coeffivients; if all of the coefficients are constant, then equation (3) is said that linear with constant coefficients.

Definition. If in equation (3) $b(x) \equiv 0$, then equation (3) is said that homogeneous otherwise it is called nonhomogeneous.

Example. Equation (4) is a second order constant coefficients homogeneous linear differential equation. Equation (5) is a fourth order variable coefficients nonhomogeneous linear differential equation.

Consider *n-th* order ordinary differential equation of the form

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0 \tag{6}$$

where F is a real function of its $(n+2)$ derivatives $x,\ y,\ \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}$.

Definition. Let $y = f(x)$ be a real function defined for all x in a real interval I and having an *n-th* derivative for all $x \in I$. The function f is called an explicit solution of the differential equation (6) on I if it satisfies the equation (6). That is, the substitution of $f(x)$ and derivatives for y and corresponding derivatives in equation (6), reduces (6) to an identity on I.

A relation $g(x, y) = 0$ is called an implicit solution of equation (6) if this relation defines one or more explicit solution of equation (6) on I.

Example. The function $f(x) = e^{3x}$ is an explicit solution of

$$\frac{dy}{dx} = 3y$$

on the interval $(-\infty, \infty)$.

Example. The relation $x^3 + y^3 - 8 = 0$ is an implicit solution of the differential equation

$$y^2 \frac{dy}{dx} = -x^2.$$

Definition. A solution of equation (6) containing n arbitrary constants is called a general solution of equation (6).

Definition. A solution of (6) obtained from a general solution of equation (6) by giving particular values to one or more of the n arbitrary constants is called a particular solution.

Example. Consider the first order differential equation

$$\frac{dy}{dx} = 3x^2 \tag{7}$$

The function $g(x) = x^3$ is a solution of equation (7). Moreover the functions $g_1(x) = x^3 + 1,\ g_2(x) = x^3 + 2, ..., g_n(x) = x^3 + c$ are solutions. The function $g_n(x) = x^3 + c$ defines the general solution of equation (7), where c is an arbitrary constant. The functions $g,\ g_1,\ g_2, ...$ are particular solutions.

Definition. A solution of equation (6) that can not be obtained from a general solution by any choice of the n arbitrary constants is called a singular solution.

Example. Consider the equation

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0 \tag{8}$$

The function $f(x) = (x + c)^2$ is a general solution of equation (8). $f_1(x) = (x + 1)^2$, $f_2(x) = (x + 2)^2, \ldots$ are particular solutions. $g(x) = 0$ is a singular solution. Because $y \equiv 0$ satisfies equation (8). But, it can not be obtained from the general solution.