CHAPTER 2. FIRST ORDER DIFFERENTIAL EQUATIONS

2.1. Seperable Equations

If a first order differential equation can be reduced to form

$$F(x)dx + G(y)dy = 0$$

then it is called seperable equation.

If an equation is seperable, then the general solution can be obtained by integrating both sides of equation

$$\int G(y)dy = -\int F(x)dx$$

So, one parameter solution is

$$g(y) = f(x) + c.$$

Example. Solve the following differential equations.
1)

$$(x-4)y^4dx - x^3(y^2-3)dy = 0$$

Solution. By dividing x^3y^4 $(x \neq 0,\ y \neq 0)$, we obtain

$$\left(\frac{1}{y^2} - 3\frac{1}{y^4}\right)dy = \left(\frac{1}{x^2} - 4\frac{1}{x^3}\right)dx.$$

Integrating above equation we obtain one parameter family of solutions

$$-\frac{1}{y} + \frac{1}{y^3} = -\frac{1}{x} + \frac{2}{x^2} + c.$$

In the seperation process, we divide given equation by x^3y^4. Note that $y = 0$ is a solution of given differential equation. But, it doesn't member of one-parameter family of solutions. So, $y = 0$ is a solution which was lost in the seperation process.
2)

$$y' = \frac{1}{2}x(1-y^2)$$

3)

$$(1+x)dy - ydx = 0$$

2.2. Homogeneous Equations

Consider the following differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

Definition. If the functions M and N in equation (1) are both homogeneous with same degree, then the differential equation (1) is called homogeneous.

Theorem. If equation (1) is a homogeneous differential equation, then the change of variables $y = vx$ transforms given equation into a seperable equation, where $v = v(x)$.

Example. Solve the following differential equations.

1)

$$xdy = \left(y + xe^{-y/x}\right) dx$$

Solution. Since the functions $M(x,y) = y + xe^{-y/x}$ and $N(x,y) = -x$ are both homogeneous with degree 1, given differential equation is homogeneous. Applying the change of variables

$$\begin{aligned} y &= vx,\ v = v(x), \\ \frac{dy}{dx} &= x\frac{dv}{dx} + v \end{aligned}$$

to given equation, we obtain a seperable differential equation

$$e^v dv = \frac{dx}{x}.$$

Integrating last equation we get

$$e^v = \ln|x| + c$$

which is the solution of seperable equation. Writing $v = \dfrac{y}{x}$, the solution of homogeneous differential equation is obtained as

$$y = x\ln(\ln|x| + c.)$$

2)

$$x^2y' = y^2 + xy - x^2$$

3)

$$2x^3ydx + (x^4 + y^4)dy = 0$$

2.3. Equations Reducible to Homogeneous Equations

Consider the differential equation of the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0, \tag{1}$$

where a_i, b_i, c_i, d_i are constants.

Theorem. *Case 1.* If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ that is, $a_1b_2 - a_2b_1 \neq 0$, then the equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \end{aligned}$$

has a solution (h, k) and the transformation

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

reduces equation (1) to the homogeneous equation

$$(a_1X + b_1Y)dX + (a_2X + b_2Y)dY = 0$$

in the variables X and Y.

Case 2. If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ that is, $a_1b_2 - a_2b_1 = 0$, then the transformation $z = a_1x + b_1y$ reduces the equation (1) to a seperable equation in the variables x and z.

Example. Solve the following differential equations.

1)

$$(x - 2y + 1)dx + (4x - 3y - 6)dy = 0$$

Solution. Here $a_1b_2 - a_2b_1 \neq 0$ and the solution of the system

$$\begin{aligned} x - 2y + 1 &= 0 \\ 4x - 3y - 6 &= 0 \end{aligned}$$

is $x = 3,\ y = 2$. So, the transformation

$$\begin{aligned} x &= X + 3 \\ y &= Y + 2 \end{aligned}$$

reduces the given equation into the homogeneous equation

$$(X - 2Y)dX + (4X - 3Y)dY = 0.$$

Applying the transformation $Y = vX,\ v = v(X)$, we obtain the following seperable equation

$$X\frac{dv}{dX} = \frac{3v^2 - 2v - 1}{4 - 3v}$$

or

$$\frac{3v-4}{3v^2-2v-1}dv = -\frac{dX}{X}.$$

Integrating last equation we obtain the solution of seperable equation as

$$X^4(3v+1)^5 = c(v-1).$$

Since $Y = vX$, the solution of homogeneous equation is

$$|3Y+X|^5 = c\,|Y-X|$$

Replacing X by $x-3$ and Y by $z-2$, we obtain the solution of given differential equation

$$|3y+x-9|^5 = c\,|y-x+1|\,.$$

2)

$$(x-y+3)dx + (2x+4y-1)dy = 0$$

3)

$$\frac{dy}{dx} = \frac{x-y+1}{x+y+3}$$